RESEARCH ARTICLE OPEN ACCESS

Approximation by Continuous Functions in the Fell Topology

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ABSTRACT

We study the approximation of multivalued functions in the Fell topology by means of single valued continuous functions or minimal upper semicontinuous compact-valued maps. We give a partial answer to Question 5.5 in [R.A. McCoy, Comparison of hyperspace and function space topologies, in: G. Di Maio, L. Holá, (Eds.), Recent Progress in Function

Keyword:- Multivalued function, Upper semicontinuity, Fell topology, Vietoris topology

I. INTRODUCTION

There exists a wide literature concerning the approximation of relations (i.e. multivalued functions) by means of singlevalued continuous functions. The first contribution, due to Cellina [4], deals with the approximation in the Hausdorff metric. More results in the same context may be found in [1,8,9,18]. The approximation of relations in the Vietoris and in the locally finite topologies has been studied in [11,12]. In particular, [11] shows that if X is a dense in itself countably paracompact normal space and $F : X \to \mathbb{R}$ is an upper semicontinuous multivalued function such that $F(x)$ is a non-empty compact interval for every $x \in X$, then F may be approximated with real-valued continuous functions in the Vietoris topology (also in the locally finite topology if X is a q-space [17]). The conditions for the approximability of F are also necessary under some suitable assumptions on the space X. Furthermore, in [12] it is proven that a countably paracompact normal space X is strongly zero-dimensional if and only if every closed subset F of $X \times \mathbb{R}$, with $pr_1(F) = X$ and $|F(x)| = 1$ at every isolated point, belongs to the closure of $C(X)$ in the Vietoris topology.

The purpose of this paper is studying approximability in the Fell topology and to give a partial answer to Question 5.5 in [14] and to Question 7 in [13]. In Section 2, Theorem 8 shows that every upper semicontinuous multivalued function F such that each value $F(x)$ is a non-empty closed interval of $\mathbb{R}$ (a singleton if x is isolated) belongs to the closure of $C(X)$ in the Fell topology. Furthermore, we provide some necessary conditions for Fell approximability with elements of $C(X)$.

Section 3 is devoted to the zero-dimensional case and shows that results analogous to [12] hold with simpler hypotheses for the approximability in the Fell topology.The last section deals with approximability by means of minimal usco maps [5,7,10] with finite range..

The main theorems are presented through some lemmas, some of which have an intrinsic interest within this topic. For instance Lemmas 7, 19 and 24, in the respective frameworks, show that only the upper topology needs to be checked.

1. Preliminaries

Let X and Y be Hausdorff spaces. If $F \subseteq X \times Y$, we define $F(x) = \{ y \in Y : (x, y) \in F \}$. In this way each subset of $X \times Y$ is viewed as a multivalued function and every multivalued function is identified with its graph.

A multivalued function F from X to Y is said to be upper semicontinuous at $x \in X$ provided that for every open subset
$V \subseteq Y$ such that $F(x) \subseteq V$ there exists a neighbourhood U of x such that $F(z) \subseteq V$ for every $z \in U$. The multivalued function F is said to be:

- upper semicontinuous, usc for short, if it is upper semicontinuous at every $x \in X$.
- cusc if it is usc and $F(x)$ is connected for every $x \in X$.
- usco if it is usc and $F(x)$ is a non-empty compact subspace of Y for every $x \in X$.
- cusco if it is cusc and usco.

We denote by $\mathrm{CL}(X \times Y)$ the set of non-empty closed subsets of $X \times Y$ and by $C(X, Y)$ the subspace consisting of continuous single-valued functions from X to Y. We write $C(X)$ for $C(X,\mathrm{R})$. $\mathrm{CL}*(X \times Y)$ indicates the subset of $\mathrm{CL}(X \times Y)$ consisting of the elements F such that $F(x) = \emptyset$ for all $x \in X$.

The following two results are well known and illustrate the connection between the semicontinuity of F and the property of F of having a closed graph.

Proposition 1. *Let Y be a regular space. Assume that F is a multivalued upper semicontinuous function from X to Y such that $F(x)$ is closed for each $x \in X$. Then F belongs to* $\mathrm{CL}(X \times Y)$.

The next proposition holds for every topological space X (e.g. see [3, p. 112]):

Proposition 2. *Let $F \in \mathrm{CL}(X \times Y)$ such that $F(X) = \mathrm{pr}_2 F$ has compact closure in Y. Then F is upper semicontinuous.*

In general, we denote the complement of E in Z by $Z \setminus E$. If $E \subseteq X \times Y$, we also indicate the complement of E in $X \times Y$ by E^c.

For every open subset W of $X \times Y$ define $W^+ = \{ F \in \mathrm{CL}(X \times Y): F \subseteq W \}$ and $W^- = \{ F \in \mathrm{CL}(X \times Y): F \cap W = \emptyset \}$.

The upper (lower) Vietoris topology on $\mathrm{CL}(X \times Y)$ is defined as the topology which has a base (subbase) consisting of sets of the form $W+$ (of the form $W-$), where W ranges over all open subsets of $X \times Y$. The Vietoris topology is the supremum of the upper and lower Vietoris topologies.

The upper Fell topology on $\mathrm{CL}(X \times Y)$ is defined as the topology which has a base consisting of sets of the form $W+$, where W ranges over open subsets of $X \times Y$ such that W^c is compact. The Fell topology is the supremum of the upper Fell topology and the lower Vietoris topology.

For more results about hyperspace topologies, see also [2,16].

II. REAL FUNCTIONS

The following statement adds an equivalent condition to a well-known theorem of Dowker–Katetov.

Lemma 3. *The following conditions on a T_1 space X are equivalent*:

(i) *X is normal and countably paracompact.*

(ii) *For every pair f, g of real-valued functions defined on X, where f is upper semicontinuous and g is lower semicontinuous and $f(x) < g(x)$ for every $x \in X$, there exists $h \in C(X)$ such that $f(x) < h(x) < g(x)$ for every $x \in X$.*

(iii) *If W is an open subset of $X \times \mathrm{R}$ such that $W(x)$ is a non-empty connected set for all $x \in X$, then there exists a function h belonging to $W^+ \cap C(X)$.*

Proof. (i) $\Leftrightarrow$ (ii) is a theorem of Dowker–Katetov [6, 5.5.20]. (ii) $\Rightarrow$ (iii) See [11, Lemma 4.1].

(iii) ⇒ (ii) Let $= \{(x, y) \in X \times \mathbb{R} : f(x) < y < g(x)\}$ W. Obviously $W(x)$ is non-empty and connected for each x. The g implies that W semicontinuity of f and is an open set. Consequently there exists a continuous real-valued function h belonging to W^+. □

If V is a family of subsets of a set Z and $E \subseteq Z$, we denote by St(E,V) the set $\bigcup\{V \in \mathcal{V}: E \cap V \neq \emptyset\}$.

The following lemma is an alternative version of [11, Lemma 4.3].

Lemma 4. *Let X be a Tychonoff space and let $F \in$ CL∗($X \times$ R) be the graph of a cusc map. Let W_0 be an open subset of $X \times$ R such that $F \subseteq W_0$. Assume that one of the following conditions holds*:

(i) *F is a cusco map.* (ii) *W_0^c is compact.*

Then there exists an open set $W \subseteq X \times$ R such that $F \subseteq W \subseteq W_0$ and $W(x)$ is connected for every $x \in X$. Furthermore, in the latter case, the open set W may be chosen in such a way that $W^+ \cap C(X,$R$) = \emptyset$.

Proof. *Case* (i) Since $F(x)$ is a compact interval, there exist an open neighbourhood A_x of x and an open interval $J_x \supset F(x)$ such that $A_x \times J_x \subseteq W_0$. Let $O_x \subseteq A_x$ be an open neighbourhood of x such that $F(z) \subseteq J_x$ for every $z \in O_x$. We are going $= \bigcup_{x\in X}(O_x$ to prove that $^W\times J_x) \subseteq W_0$ is the required open set.

Let $z \in X$. Then $W(z) = \bigcup\{ J_x: z \in O_x \}$ is connected because is the union of a collection of intervals containing $F(z)$.

Case (ii) Let (a,b) be an open bounded interval such that $\mathrm{pr}_2(W_0^c) \subseteq (a,b)$, so that $X \times ($R$ \setminus [a,b\,]) \subseteq W_0$. If we define $G(x) = F(x) \cap [a,b\,]$, we obtain that G is a cusc multivalued function which assumes (possibly empty) compact values. Arguing as in case (i), there exist an open neighbourhood N_x of x and an open interval J_x, with $J_x = \emptyset$ whenever $G(x) = \emptyset$, such that $G(z) \subseteq J_x$ for every $z \in N_x$ and $N_x \times J_x \subseteq W_0$. For every $x \in X$ put

$$I_x = \mathrm{St}\big(F(x), \{J_x, (-\infty, a), (b, +\infty)\}\big).$$

Then $F(x) \subseteq I_x$ and $N_x \times I_x \subseteq W_0$. Let $O_x \subseteq N_x$ be an open neighbourhood of x such that $F(z) \subseteq I_x$ for every $z \in O_x$. Take the set $W = \bigcup_{x\in X}(O_x \times I_x) \cup (K^c \times$ R$)$, where $K = \mathrm{pr}_1(W_0^c)$. Clearly $F \subseteq W \subseteq W_0$ and it is easy to show that $W(z)$ is connected for every $z \in X$.

Since K is normal and countably paracompact, by Lemma 3 there exists $h \in C(K,$R$)$ such that $(x,h(x)) \in W$ for every $x \in K$. Any continuous extension of h from the compact subspace K to X satisfies the last requirement. Notice that complete regularity in Lemma 4 is required only for the last statement of case (ii).

Lemma 5. *Let X be a regular space and take $F \in$ CL($X \times$ R) such that $|F(x)| \leqslant 1$ for every isolated point $x \in X$. If $F \in \bigcap_{i=1}^n W_i^-$, where every W_i is an open subset of $X \times$ R, then for each i there exist a non-empty open subset U_i of X and a non- empty open interval V_i such that $U_i \times V_i \subseteq W_i$, and if $i = j$ one of the following conditions is satisfied*:

- $\overline{U_i} \cap \overline{U_j} = \emptyset$
- *U_i is a singleton and $U_i \times V_i = U_j \times V_j$.*

Proof. For each i choose a point $(x_i, y_i) \in W_i \cap F$. Let I_1 denote the set of indexes i such that x_i is isolated and let I_2 be the complement of I_1 in the set of indexes i. If $x_i \in I_1$ put $U_i = \{ x_i \}$ and let V_i be an open connected neighbourhood of y_i such that $\{x_i\} \times \overline{V_i} \subseteq W_i$. Notice that if $x_i = x_j$ for $i, j \in I_1$, then $y_i = y_j$. In this case we replace V_i with $\bigcap\{V_j: x_j = x_i\}$.

The points $(x_i, y_i) \in W_i$ with $i \in I_2$ may be replaced by points $(x_i', y_i') \in W_i$ in such a way that the first coordinates are distinct (use the fact that the points x_i for $i \in I_2$ are limit points). Finally, by using the regularity of X, we can choose open neighbourhoods U_i of x_i and V_i of y_i satisfying the requirements, with the caution that $x_j \notin U_i$ if $i \in I_2$ and $j \in I_1$. □

The following result provides a required condition for a multivalued function to be approximated in the lower Vietoris topology.

Proposition 6. *Let X and Y be Hausdorff spaces and let Z $\subseteq$ CL($X \times Y$) such that $|Z(x)| \leqslant 1$ for each isolated point $x \in X$ and for each $Z \in$ Z. If F belongs to the closure of Z in the lower Vietoris topology, then $|F(x)| \leqslant 1$ for each isolated point $x \in X$.*

Proof. Let $x \in X$ be an isolated point. If $V_1, V_2 \subseteq Y$ are disjoint open subsets such that $F(x) \cap V_i = \emptyset$, then $(\{x\} \times V_1)^- \cap (\{x\} \times V_2)^- \cap$ Z $= \emptyset$

The following lemma suggests that only the approximation in the upper topology is to be checked.

Lemma 7. *Let X be a Tychonoff space. Let W be an open subset of X × R such that W (x) is connected for every x. Suppose that*

$W^+ \cap C(X) \neq \emptyset$. *If* $W_1, \ldots, W_n$ *are non-empty open subsets of W such that*

$pr_1(\overline{W_i}) \cap pr_1(\overline{W_j}) = \emptyset$ *whenever* $i \neq j$, *then* $C(X) \cap W^+ \cap \bigcap_{i=1}^n W_i^- \neq \emptyset$.

Proof. For each i choose a non-empty open subset U_i of X and a non-empty bounded open interval V_i in such a way that $U_i \times V_i \subseteq W_i$ and $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $i \neq j$. Notice that $\overline{V_i} \subseteq W(x)$ for every $x \in U_i$. Let $f \in W^+ \cap C(X)$ and $x_i \in U_i$ for each i. Then there exist an open interval J_i containing $\{ f(x_i)\} \cup V_i$ and an open neighbourhood G_i of x_i such that $G_i \subseteq U_i$ and $G_i \times J_i \subseteq W$ (here use that $\{ f(x_i)\} \cup V_i$ is contained in a continuum of $W(x_i)$). By using the continuity of f it is not restrictive to suppose that $f(G_i) \subseteq J_i$. For each i take a point $t_i \in V_i$ and a continuous function $\lambda_i : X \to [0,1]$ such that $\lambda_i(x_i) = 1$ and $\lambda_i(x) = 0$ for every $x \notin G_i$. The required function is:

$$g = \left(1 - \sum_{i=1}^{n} \lambda_i\right) f + \sum_{i=1}^{n} t_i \lambda_i . \qquad \square$$

We are ready to state the first theorem.

Theorem 8. *Let X be a Tychonoff space. If* $F \in CL_*(X \times R)$ *is the graph of a cusc map which maps isolated points into singletons, then F belongs to the closure of C(X) in the Fell topology.*

Proof. Let $W_0, W_1, \ldots, W_n$ be open subsets of $X \times R$ such that $F \in W_0^+ \cap \bigcap_{i=1}^n W_i^-$, where W_0^c is compact. Choose an open subset W of $X \times R$ satisfying the conditions of Lemma 4. By applying Lemma 5 to F and $W_i \cap W$, we can choose m distinct non-empty open sets W and $U_1 \times V_1, \ldots, U_k \times V_k$ in such a way that $W^+ \cap \bigcap_{k=1}^m (U_k \times V_k)^- \subseteq \bigcap_{i=1}^n (W \cap W_i)^-$ and $U_m \times V_m$ satisfy the conditions of Lemma 7, we obtain that $\overline{U_h} \cap \overline{U_k} = \emptyset$ if $h \neq k$ and $C(X) \cap W^+ \cap \bigcap_{k=1}^m (U_k \times V_k)^- \neq \emptyset$. Since the ... Consequently, $C(X) \cap W_0^+ \cap \bigcap_{i=1}^n W_i^- \neq \emptyset$. $\square$

Notice that without additional hypotheses on X (countable paracompactness and normality, see [11]) even a cusco map which maps isolated points into singletons is not necessarily approximable by continuous functions in the Vietoris topology.

Example 9. Let S be an uncountable set and let p denote the filter of co-countable subsets of S. Consider the topological space $S \cup \{ p \}$, where all points of S are isolated and a local base at p traces the elements of p on S. Let $N \cup \{\omega\}$ denote the one-point compactification of the discrete countable space N. Let $X = (S \cup \{ p \}) \times (N \cup \{\omega\}) \setminus (p,\omega)$. The closed subsets $S \times \{\omega\}$ and $\{ p \} \times [n,\omega)$ for every $n \in N$ show that X is neither normal nor countably paracompact.

Take an uncountable subset $A \subseteq S$ such that $S \setminus A$ is uncountable and denote by g the characteristic function of A. Consider the cusco multivalued function so defined:

$$F(x,\alpha) = \begin{cases} g(x) & \text{for all } x \in S \text{ and } \alpha \in \mathbb{N} \cup \{\omega\} \\ [0,1] & \text{if } x = p \text{ and } \alpha \in \mathbb{N}, \end{cases}$$

Theorem 8 says that F is approximated by continuous functions in the Fell topology. In order to show that F cannot be approximated by continuous functions in the Vietoris topology, consider the open neighbourhood H of F so defined:

$$H = \left(A \times \{\omega\} \times \left(\frac{1}{2}, +\infty\right)\right) \cup \left((S \setminus A) \times \{\omega\} \times \left(-\infty, \frac{1}{2}\right)\right) \cup ((S \cup \{p\}) \times \mathbb{N} \times \mathbb{R}).$$

Assume there exists $f \in C(X) \cap H^+$ and put $y_n = f(p,n)$ for every $n \in N$. A standard argument of cardinality shows that there exists a co-countable subset $T \subseteq S$ such that $f(x,n) = y_n$ for every $x \in T$ and $n \in N$. Take any $x_1 \in A \cap T$ and $x_2 \in (S \setminus A) \cap T$. The continuity of f at the points (x_1,ω) and (x_2,ω) implies that there exists $k \in N$ such that $y_k = f$

$(x_1, k) > \frac{1}{2}$and $y_k = f(x_2, k) < \frac{1}{2}$, a contradiction. Consequently F does not belong to the closure of $C(X)$ in the upper Vietoris topology.

Remark 10. A cusc map $F \in$ CL($X \times$ R) may belong to the closure of $C(X)$ in the Fell topology even if the set { x: $F(x) = \emptyset$} is a non-empty open set. For instance, consider the map| x| > 1, $F(x) =$ [1, +∞) if | x| = 1, $F(x) = \emptyset$ if | x| < 1. F on the real interval $X =$ [−2,2] defined as follows: $F(x) = \frac{1}{x}$| if

In some situations, a multivalued map which belongs to the closure of $C(X)$ in the Fell topology must be necessarily cusc at any point x such that $F(x) = \emptyset$.

Proposition 11. *Let X be a locally connected locally compact regular space and let F* $\in$ CL($X \times$ R). *If F is in the closure of* $C(X)$ *in the Fell topology, then F is a cusc map at every point x such that* $F(x) = \emptyset$.

Proof. Argue as in [11, Lemmas 3.2, 3.3] by choosing the neighbourhood U of x as a connected open set with compact closure.

None of the two relevant properties of the space X in Proposition 11 can be removed.

Example 12.

- *Compact case.* Let $X = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\}$ with the usual topology and let $F(\frac{1}{n}) = 1$ and $F(0) = \{0,1\}$. Then F belongs to the closure of $C(X)$ in the Vietoris topology (cf. [12, Theorem 3.2]).
- *Locally connected case.* Let $X = J(\aleph_0)$ be the hedgehog with $\aleph_0$-many spines. Denote by 0 the centre of the hedgehog and by I_n the nth spine. Define $F(x) = 1$ for every $x = 0$ and $F(0) = \{0,1\}$. Then F belongs to the closure of $C(X)$ in the Fell topology (although not in the Vietoris topology, see [11, Lemmas 3.2, 3.3]). Indeed, consider $F \in (K^c)^+$, where K is a compact subset of $X \times [0,1]$. If $K_n = K \cap (I_n \times [0,1])$ and $r_n = \max \mathrm{pr}_1(K_n)$, then for every $\varepsilon > 0$ there exists j such that $r_j < \varepsilon$. Put $E_j = \{ x \in I_j : x \in (\varepsilon, 2\varepsilon)\}$. An approximating continuous function may be defined in such a way that it assumes the value 0 at some point of E_j and the value 1 at every point of $X \setminus E_j$.

If we want an example which is not upper semicontinuous, it is enough to change the definition of F in both cases by putting $F(x) = \frac{1}{x}$ for $x = 0$.

A multivalued map belonging to CL($X \times$ R) is said to be *bounded on* $E \subseteq X$ if $F(E) = \bigcup_{x \in E} F(x)$ is bounded [15]. If F is bounded (on X), then F is upper semicontinuous (see Proposition 2). The following Proposition is proved in [2, Proposition 6.2.11].

Proposition 13. *Let F* $\in$ CL($X \times Y$) *be a compact valued usc map. Then F(E) is compact for each compact subspace E of X.*

Propositions 2 and 13 imply the following:

Corollary 14. *Let X be a compact space and let F* $\in$ CL($X \times Y$) *be a compact valued map. Then F is usc if and only if F is bounded.*

Proposition 15. *Let X be a continuum and let F* $\in$ CL($X \times$ R) *be a bounded map. If F belongs to the closure of* $C(X)$ *in the Fell topology, then*:

(i) $F(x) = \emptyset$ *for every* $x \in X$;

(ii) *the graph of F is a connected subset of* $X \times$ R (*hence* $F(X)$ *is a connected subset of* R).

Proof. Consider a bounded interval (a,b) such that $F(X) \subseteq (a,b)$.

(i) By way of contradiction, suppose there exists a point $h \in X$ such that $F(h) = \emptyset$ and let $V = X \setminus \{h\}$. Put

$$\mathcal{K} = [((\{h\} \times [a,b]) \cup (X \times \{a,b\}))^c]^+ \cap [V \times (a,b)]^-.$$

Then K is an open set in the Fell topology, $F \in$ K, K $\cap$ $C(X) = \emptyset$, a contradiction.

(ii) By way of contradiction, assume that the compact subset F is the union of two disjoint non-empty closed subsets F_1 and F_2. Then F_1 and F_2 are compact subsets of $X \times (a,b)$. Therefore there exist disjoint open subsets W_1, W_2 of $X \times (a,b)$ such that $F_1 \subseteq W_1$ and $F_2 \subseteq W_2$. Put

$\mathcal{L} = W_1^- \cap W_2^- \cap \left[(X \times [a,b]^c) \cup W_1 \cup W_2 \right]^+$.

Then L is an open subset in the Fell topology, $F \in$ L, L $\cap$ $C(X)$ =∅ , a contradiction.

Remark 16. The function F of Remark 10 shows that the boundedness of F is necessary in Proposition 15. Even if F satisfies the hypotheses of Theorem 8, the graph of$x$ = 0, F(0) = R. F is not necessarily connected. For example, consider $F(x) = \frac{1}{x}$| if | x| 1 and

Proposition 17. *Let X be a continuum. If F* ∈ CL(X × R) *belongs to the closure of* $C(X)$ *in the Fell topology, then* $F(X)$ *is connected.*

Proof. By contradiction, assume there exist real numbers $a < b < c$ such that $a \in F(X)$, $c \in F(X)$, $b \notin F(X)$. Then F belongs to

$\mathcal{H} = \left[(X \times \{b\})^c \right]^+ \cap \left[X \times (-\infty, b) \right]^- \cap \left[X \times (b, +\infty) \right]^-$.

Since X is connected, $\mathcal{H} \cap C(X) = \emptyset$. □

Remark 18. Let X be a locally compact locally connected space. Let $F \in$ CL(X × R) belong to the closure of $C(X)$ in the Fell topology. If $F(x)$ =∅ and $F(x) \subseteq (a,b)$, then either x is an isolated point or x is a limit point for the set $\{y \in X:\ F(y) \neq \emptyset\}$. To prove this assertion, use a construction similar to the one made in Proposition 15 by putting

$\mathcal{K} = \left[\left((\overline{U} \times \{a,b\}) \cup (\{z\} \times [a,b]) \right)^c \right]^+ \cap \left[U \times (a,b) \right]^-$,

where U is a connected neighbourhood of the non-isolated point x such that U is compact, $F(y)$ = ∅ for every $y \in U \setminus \{x\}$ and z is a point of $U \setminus \{x\}$.

The zero-dimensional case

We provide an analogous version of Lemma 7 in the zero-dimensional case, that is if X has a base of clopen sets.

Lemma 19. *Let X be a zero-dimensional space and let Y be a Hausdorff space. Let W be an open subset of X × Y such that* $W^+ \cap C(X,Y) \neq \emptyset$. *If* $W_1,\dots,W_n$ *are non-empty open subsets of W such that* $\mathrm{pr}_1(W_i) \cap \mathrm{pr}_1(W_j)$ =∅ *whenever i = j, then* $C(X,Y) \cap W^+ \cap \bigcap_{i=1}^n W_i^- \neq \emptyset$.

Proof. Let $g \in C(X,Y) \cap W^+$. For each $i \in \{1,\dots,n\}$ choose a clopen set $U_i \subseteq X$ and a point $y_i \in Y$ such that $U_i \times \{y_i\} \subseteq W_i$. The required element of $C(X,Y)$ is the function f so defined: $f(x) = y_i$ if $x \in U_i$, $f(x) = g(x)$ if $x \notin \bigcup_{i=1}^n U_i$.

Lemmas 5 and 19 imply the following:

Corollary 20. *Let X be a zero-dimensional space and let F* ∈ CL(X × Y) *such that* | $F(x) \leqslant 1$ | *for every isolated point x* ∈ *X. If F belongs to the closure of* $C(X,Y)$ *in the upper Fell topology, then F belongs to the closure of* $C(X,Y)$ *in the Fell topology.*

Remark 21. If Y is a compact space, then $[(\{x\} \times Y)^c]^+ \cap \mathrm{CL}_*(X \times Y) = \emptyset$. Consequently $\mathrm{CL}_*(X \times Y)$ is closed in CL(X × Y) in the upper Fell topology.

Theorem 22. *Let X be a zero-dimensional space and let Y be a Hausdorff space.*

- *If Y is compact, then the closure of* $C(X,Y)$ *in* CL(X × Y) *with the Fell topology is the set of all F* ∈ $\mathrm{CL}_*(X \times Y)$ *which take a single value at every isolated point x* ∈ *X.*
- *If Y is not compact, then the closure of* $C(X,Y)$ *in* CL(X × Y) *with the Fell topology is the set of all F* ∈ CL(X × Y) *such that* | $F(x) \leqslant 1$| *for every isolated point x* ∈ *X.*

Proof. By Proposition 6 and Remark 21, in both cases the conditions on F are necessary.

By Corollary 20 it suffices to show that F belongs to the closure of $C(X,Y)$ in the upper Fell topology. Assume that $F \in W^+$, where W^c is a compact subset of $X \times Y$.

- Suppose Y is compact. Put $K = \mathrm{pr}_1 W^c$. For every $x \in K$ choose a clopen neighbourhood U_x of x in X and a point $y_x \in F(x)$ such that $U_x \times \{y_x\} \subseteq W$. Since K is compact, choose an irreducible finite subcovering $U_{x1},\dots,U_{xn}$ of the clopen covering $\{U_x: x \in K\}$. Let $V_i = U_{xi} \setminus {}_{j<i} U_{xj}$. Then $\{V_i: 1\ \ i\ \ n\} \in$ is a disjoint covering of∩ K consisting of non-empty clopen sets. If y denotes any point of Y , the required function f $C(X,Y)$ W^+ is defined by

$(x) = \{ \; yx_i$ if $x \in V_i$, y if $x \in X \setminus \bigcup\{ \; i$
$V : 1 \; i \; n \}$.

- If Y is not compact, choose any $\bar{y} \in (Y \setminus \mathrm{pr}_2 \, W^c)$. Then the required function in $C(X, Y) \cap W+$ is the constant function $g(x) = \bar{y}$. □

Theorem 23. *Let X be a locally compact topological space. The following conditions are equivalent*:

(i) *X is zero-dimensional.*

(ii) *For every compact space Y , the closure of C(X, Y) in CL(X × Y) with the Fell topology consists of all F ∈* CL∗(X × Y) *which map isolated points into singletons.*

(iii) *The closure of C(X,* [0,1]) *in CL(X ×* [0,1]) *with the Fell topology consists of all F ∈* CL∗(X × [0,1]) *which map isolated points into singletons.*

(iv) *The closure of C(X,* {0,1}) *in CL(X ×* {0,1}) *with the Fell topology consists of all F ∈* CL∗(X × {0,1}) *which map isolated points into singletons.*

Proof. Theorem 22 shows that it suffices to prove (iii) ⇒ (i) and (iv) ⇒ (i).

Let p denote a non-isolated point and take an open neighbourhood U of p such that $\bar{U}$ is compact. We are going to prove that U contains a clopen neighbourhood of p.

Choose an open neighbourhood V of p such that $\bar{V} \subseteq U$ and consider the following multivalued function

$$F = (\bar{V} \times \{0\}) \cup ((X \setminus V) \times \{1\}).$$

Since $\bar{V} \setminus V$ contains no isolated points, the multivalued function F maps isolated points into singletons. Notice that F may be viewed as an element both of CL∗(X × [0,1]) and of CL∗(X × {0,1}).

(iii) ⇒ (i). Choose a positive real number $\varepsilon \leqslant \frac{1}{2}$ and consider the following compact subset of X ×[0,1]

$$K = (\{p\} \times [\varepsilon, 1]) \cup (\bar{U} \times [\varepsilon, 1-\varepsilon]) \cup ((\bar{U} \setminus U) \times [0, 1-\varepsilon]).$$

Then (K^c)+ is a Fell-neighbourhood of F. Consequently there exists $f \in (K^c)^+ \cap C(X, [0,1])$. Set $Z = \{x \in U: f(x) < \varepsilon\}$. Notice that $p \in Z$ and $\bar{U} \setminus Z = \{x \in \bar{U}: f(x) > 1-\varepsilon\}$. Obviously Z is an open set. It is also closed because $X \setminus Z = (X \setminus \bar{U}) \cup \{x \in X: f(x) > 1-\varepsilon\}$ is open.

(iv) ⇒ (i). Consider the following compact subset of X ×{0,1} :

$$L = (\{p\} \times \{1\}) \cup ((\bar{U} \setminus U) \times \{0\}).$$

Since (L^c)+ is a Fell-neighbourhood of F, there exists $f \in (L^c)^+ \cap C(X, \{0,1\})$. The set $Z = \{x \in U: f(x) = 0\}$ is an open neighbourhood of p. Since $X \setminus Z = (X \setminus \bar{U}) \cup \{x \in X: f(x) = 1\}$, it follows that Z is also closed. □

On minimal usco maps

Let H ⊆ CL∗(X × Y) denote the set of usco maps. A standard application of Zorn's lemma ensures that, if H is ordered by set inclusion, then every F ∈ H contains a minimal element G ∈ H. The set of minimal elements of H is denoted by $M(X, Y)$. Each element of $M(X, Y)$ is called a minimal usco map. The following statement is analogous to Lemmas 7 and 19.

Lemma 24. *Let X be a regular space and let Y be a Hausdorff space. Let W be an open subset of X × Y such that* $W^+ \cap M(X,Y) \neq \emptyset$. *If* $W_1,\dots, W_n$ *are non-empty open subsets of W such that* $\mathrm{pr}_1(W_i) \cap \mathrm{pr}_1(W_j) = \emptyset$ *whenever i = j, then* $M(X, Y) \cap W^+ \cap \bigcap_{i=1}^{n} W_i^- = \emptyset$.

Proof. Let $F \in M(X,Y) \cap W^+$. For every $i \in \{1,\dots,n\}$.

choose an open set $U_i \subseteq X$ and a point $y_i \in Y$ such that $U_i \times \{ y_i \} \subseteq W_i$ and $U_i \cap U_j = \emptyset$ if $i = j$. Consider the multifunction $L \in \mathrm{CL}*(X \times Y)$ defined as follows:

$\in i;^{+} \cap \bigcap_{i=1}^{n} W_i^{-}$. If $G \subseteq L$ is a minimal usco map,We denote by $D(X, Y) \subseteq \mathrm{CL}(X \times Y)$ the set consisting of the elements of the form $f \ C(f)$, where f is a single valued function which is continuous at every point x of a dense subset $C(f)$ [7,10].

$$L(x) = \begin{cases} \{y_i\} & \text{if } x \quad U \\ \{y_i\} \cup F(x) & \text{if } x \in (\overline{U_i} \setminus U_i); \\ F(x) & \text{if } x \notin \ {}_{i=1}^{n} \overline{U_i} \end{cases}$$

By Proposition 1, $L \in \mathrm{CL}^*(X \times Y)$. Clearly L is an usco map belonging to W then $G \in W^{+} \cap \ {}_{i=1}^{n} W_i^{-}$ because $G(U_i) = \{y_i\}$.

Consider the subset $\mathrm{DFin}(X \times Y)$ of $\mathrm{CL}*(X \times Y)$ consisting of multivalued functions of the form

$$G = \bigcup_{i=1}^{n} (\overline{V_i} \times \{y_i\})$$

of where X. One can easily show that DFiny_i are elements of Y and V_i(are open subsets of $X \times Y) \subseteq M(X, Y) \cap$ DX(Xsuch that, Y).$V_i \cap V_j =$ $\emptyset$ for $i = j$ and ${}^{n}{}_{i}$=1 V_i is a dense subset

Remark 25. In the proof of Lemma 24, assume that $F \in \mathrm{DFin}(X \times Y)$. Then the minimal usco map G belongs to $\mathrm{DFin}(X \times Y)$.

Remark 26. Let $x \in X$ be an isolated point. If $G \in M(X, Y)$, then $|G(x)| = 1$. Consequently, if F belongs to the closure of $M(X, Y)$ in the lower Vietoris topology, then $|F(x) \leqslant| 1$ (see Proposition 6).

Lemmas 5 and 24 and Remark 25 imply the following:

Corollary 27. *Let X be a regular topological space and let Y be a Hausdorff space. Let $F \in \mathrm{CL}(X \times Y)$ such that $|F(x) \leqslant 1|$ for every isolated point $x \in X$. If F belongs to the closure of* $\mathrm{DFin}(X \times Y)$ *in the upper Fell topology, then F belongs to the closure of* $\mathrm{DFin}(X \times Y)$ *in the Fell topology.*

Theorem 28. *Let X be a regular topological space and let Y be a Hausdorff space. The closure of* $\mathrm{DFin}(X \times Y)$ *in the Fell topology of $CL(X \times Y)$ coincides with the set* E *described in the following cases*:

- *if Y is compact, then* E *consists of all* $F \in \mathrm{CL}*(1$ $X \times Y)$ *such that* $|F(x)| =$ *if x is an isolated point.*
- *if Y fails to be compact, then* E *consists of all* $F \in CL(X \times Y)$ *such that* $|F(x) \leqslant 1|$ *if x is an isolated point.*

Proof. In both cases, by Remarks 26 and 21, the conditions on E are required.

Choose a non-empty open subset W of $X \times Y$ such that W^c is compact.

- Suppose that Y is compact and that W + contains an element $F \in \mathrm{CL}*(X \times Y)$ such that $|F(x)| = 1$ for each isolated point $x \in X$. Let $K = \mathrm{pr}_1(W^c)$. For every $x \in K$ take an open neighbourhood U_x of x in X and a point $y_x \in F(x)$ such that $U_x \times \{ y_x \} \subseteq W$. Since K is compact, choose an irreducible finite subcover $U_{x1}, \dots, U_{xn}$ of the open cover $\{ U_x : x \in K \}$.

Let us set $V_i = U_{xi} \setminus \bigcup_{<j_i} U_{xj}$ and $V_0 = X \setminus \bigcup_{i=1}^{n} V_i$. Choose an arbitrary element $y_0 \in Y$ and define

$$G = \bigcup_{i}^{n} (\overline{V_i} \times \{y_i\}).$$

$i=0$

Then G belongs to $\mathrm{DFin}(X \times Y) \cap W^+$. By Corollary 27 the element F belongs to the closure of E in the Fell topology.

- Suppose that Y fails to be compact and choose $y \notin \mathrm{pr}_2(W^c)$. Then $X \times \{y\}$ belongs to W^+ and consequently $\mathrm{DFin}(X \times Y)$ is dense in $\mathrm{CL}(X \times Y)$ with the upper Fell topology. The conclusion follows by applying Corollary 27.

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