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Essential Subspace Attentiveness Circumstances for the Even Double Minkowski Delinquent

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ABSTRACT

We prove tight subspace concentration inequalities for the Received 28 April [2017] dual curvature measures $Cq(K,\cdot)$ of an n-dimensional origin symmetric convex body for $q \geq n+1$. This supplements former results obtained in the range $q \leq n$.

Keywords:
Dual curvature measure, Cone-volume measure, Surface area measure L_p-Minkowski problem, Logarithmic Minkowski, problem Dual Brunn–Minkowski theory.

I. INTRODUCTION

Let K^n denote the set of convex bodies in R^n, i.e., the family of all non-empty convex and compact subsets $K \subset R^n$. The set of convex bodies having the origin as an interior point is denoted by K_o^n and the subset of origin-symmetric convex bodies, i.e., those sets $K \in K_o^n$ satisfying $K = -K$, is denoted by K_e^n. For $\boldsymbol{x}, \boldsymbol{y} \in R^n$, let $\boldsymbol{x}, \boldsymbol{y}$ denote

the standard inner product and $|\boldsymbol{x}| = \boldsymbol{x},\boldsymbol{x}\{$ the$\in$ Euclidean$|$ $| \leq$ norm.$\}$ We write B_n for the n-dimensional Euclidean unit ball, i.e., $B_n = \boldsymbol{x}\ R^n : \boldsymbol{x}$ 1 and $S^{n-1} = \partial B_n$, where ∂K is the set of boundary points of $K \in K^n$. The k-dimensional Hausdorff-measure will be denoted by $H^k(\cdot)$ and instead of $H^n(\cdot)$ we will also write vol$(\cdot)$ for the n-dimensional volume. For a k-dimensional set $S \subset R^n$ we also write $\mathrm{vol}_k(S)$ instead of $H^k(S)$.

At the heart of the Brunn–Minkowski theory is the study of the volume functional with respect to the Minkowski addition of convex bodies. This leads to the theory of mixed volumes and, in particular, to the quermassintegrals $W_i(K)$ of a convex body $K \in K^n$. The latter may be defined via the classical Steiner formula, expressing the volume of the Minkowski sum of K and λB_n, i.e., the volume of the parallel body of K at distance λ as a polynomial in λ (cf., e.g., [48, Sect. 4.2])

$$\mathrm{vol}(K+\lambda B_n) = \lambda^i \; W_i(K). \quad (1.1)\; {}_n\, n \quad i \; {}_{i=0}$$

A more direct geometric interpretation is given by Kubota's integral formula (cf., e.g., [48, Subsect. 5.3.2]), showing that they are — up to some constants — the means of the volumes of projections

$$W_{n-i}(K) = \mathrm{vol}\_\_\_\_\_\_\mathrm{vol}(_i(BB^n{}_i)) \; \mathrm{vol}_i(K|L)\mathrm{d}L, \quad i = 1,\ldots,n, \qquad (1.2)$$
$$G(n,i)$$

where integration is taken with respect to the rotation-invariant probability measure on the Grassmannian $G(n, i)$ of all i-dimensional linear subspaces and $K|L$ denotes the image of the orthogonal projection of K onto L.

A local version of the Steiner formula above leads to two important series of geometric measures, the area measures $S_i(K,\cdot)$ and the curvature measures $C_i(K,\cdot)$, $i = 0, \ldots, n-1$, of a convex body K. Here we will only briefly describe the area measures, since with respect to characterization problems of geometric measures they form the "primal" counterpart to the dual curvature measures we are interested in.

To this end, for $\omega \subseteq S^{n-1}$ we denote by $\nu_K^{-1}(\omega) \subseteq \partial K$ the set of all boundary points of K having an outer unit normal in ω. When K is a smooth convex body, it is the inverse of the Gauss map assigning to a boundary point of K its unique outer unit normal. Moreover, for $\boldsymbol{x} \in R^n$, let $r_K(\boldsymbol{x}) \in K$ be the point in K closest to $\boldsymbol{x}$. Then for a Borel set $\omega \subseteq S^{n-1}$ and $\lambda > 0$ we consider the local parallel body

$$B_K(\lambda,\omega) = \boldsymbol{x} \in R^n : 0 < |\boldsymbol{x} - r_K(\boldsymbol{x})| \leq \lambda \text{ and } r_K(\boldsymbol{x}) \in \nu_K^{-1}(\omega) \quad . \qquad (1.3)$$

The local Steiner formula expresses the volume of $B_K(\lambda, \omega)$ as a polynomial in λ. Its coefficients are (up to constants depending on i, n) the area measures (cf., e.g., [48, Sect. 4.2])

$$\mathrm{vol}(B_K(\lambda,\omega)) = \frac{1}{n}\sum_{i=1}^{n} \binom{n}{i} \lambda^i\, \mathrm{S}_{n-i}(K,\omega). \qquad (1.4)$$

$\mathrm{S}_{n-1}(K, \cdot)$ is also known as the surface area measure of K. The area measures may also be regarded as the (right hand side) differentials of the quermassintegrals, since for $L \in \mathcal{K}^n$

$$\lim_{\epsilon \downarrow 0} \frac{\mathrm{W}_{n-1-i}(K+\epsilon L) - \mathrm{W}_{n-1-i}(K)}{\epsilon} = \int_{\mathbb{S}^{n-1}} h_L(\boldsymbol{u})\,\mathrm{dS}_i(K,\boldsymbol{u}). \qquad (1.5)$$

Here $h_L(\cdot)$ denotes the support function of L (cf. Section 2). Also observe that $\mathrm{S}_i(K, S^{n-1}) = n\, \mathrm{W}_{n-i}(K)$, $i = 0, \ldots, n-1$.

A cornerstone of the Brunn–Minkowski theory is to characterize the area measures $\mathrm{S}_i(K, \cdot)$, $i \in \{1, \ldots, n-1\}$, among the finite Borel measures on the sphere. Today this problem is known as the *Minkowski–Christoffel* problem, since for $i = n - 1$ it is the classical Minkowski problem and for $i = 1$ it is the Christoffel problem. We refer to [48, Chapter 8] for more information and references.

There are two far-reaching extensions of the classical Brunn–Minkowski theory, both arising basically by replacing the classical Minkowski-addition by another additive operation (cf. [22,23]). The first one is the L_p addition introduced by Firey (see, e.g., [18]), which leads to the rich and emerging *L_p-Brunn–Minkowski theory* for which we refer to [48, Sect. 9.1, 9.2]. The second one, introduced by Lutwak [36,37], is based on the radial addition $\tilde{+}$, where $\boldsymbol{x} \tilde{+} \boldsymbol{y} = \boldsymbol{x} + \boldsymbol{y}$, if $\boldsymbol{x}, \boldsymbol{y}$ are linearly dependent, and $\mathbf{0}$ otherwise. Considering the volume of radial sums leads to the *dual Brunn–Minkowski theory* (cf. [48, Sect. 9.3]) with dual mixed volumes, and, in particular, also with dual quermassintegrals $\widetilde{\mathrm{W}}_i(K)$, arising via a dual Steiner formula (cf. (1.1))

$$\mathrm{vol}(K \tilde{+} \lambda B_n) = \sum_{i=0}^{n} \binom{n}{i} \lambda^i\, \widetilde{\mathrm{W}}_i(K).$$

Here $K \tilde{+} \lambda B_n$ is the dual outer parallel body of K. In general the radial addition of two convex sets is not a convex set, but the radial addition of two star bodies is again a star body. This is one of the features of the dual Brunn–Minkowski theory which makes it so useful. The celebrated solution of the Busemann–Petty problem is amongst the recent successes of the dual Brunn–Minkowski theory, cf. [19,24,52], and it also has connections and applications to integral geometry, Minkowski geometry and the local theory of Banach spaces.

In analogy to Kubota's formula (1.2), the dual quermassintegrals $\widetilde{\mathrm{W}}_i(K)$ admit the following integral geometric representation as the means of the volumes of sections (cf. [48, Sect. 9.3])

$$\widetilde{\mathrm{W}}_{n-i}(K) = \frac{\mathrm{vol}(B_n)}{\mathrm{vol}_i(B_n^i)} \int_{G(n,i)} \mathrm{vol}_i(K \cap L)\,\mathrm{d}L, \qquad i = 1,\ldots,n.$$

There are many more "dualities" between the classical and dual theory, but there were no dual geometric measures corresponding to the area and curvature measures. This missing link was recently established in the ground-breaking paper [31] by Huang, Lutwak, Yang and Zhang. Let ρ_K be the radial function (cf. Section 2) of a convex body $K \in \mathcal{K}_o^n$.

Analogous to (1.3), we consider for a Borel set $\eta \subseteq S^{n-1}$ and $\lambda > 0$ the set

$A_K(\lambda,\eta) = \{\mathbf{0}\} \cup \{\cup\{\mathbf{x} \in\in K \setminus \{\setminus \mathbf{0}\} : |\rho_{K}-(\mathbf{x})\mathbf{x} \in v_{K-}| \leq_1(\eta)\}$ $\in$ $\}$

$\mathbf{x}\ \mathrm{R}^n$ $K : \mathbf{x}\ \rho_K(\mathbf{x})\mathbf{x}$ λ and $\rho_K(\mathbf{x})\mathbf{x}\ v_K^{-1}(\eta)$.

Then there also exists a local Steiner type formula of these local dual parallel sets [31, Theorem 3.1] (cf. (1.4))

$$\mathrm{vol}(A_K(\lambda,\eta)) = \lambda^i \mathrm{C}_{n-i}(K,\eta).\ n\ n\ i\ i{=}0$$

$\mathrm{C}_i(K,\cdot)$ is called the *ith* $^{dual}\cdot$ *curvature measure* and they are the counterparts to the curvature measures $\mathrm{C}_i(K,)$ within the dual Brunn–Minkowski theory. Observe that $\mathrm{C}_i(K, S^{n-1}) = \mathrm{W}_{n-i}(K)$. As the area measures (cf. (1.5)), the dual curvature measures may also be considered as differentials of the dual quermassintegrals, even in a stronger form (see [31, Section 4]). We want to point out that there are also dual area measures corresponding to the area measures in the classical theory (see [31]).

Huang, Lutwak, Yang and Zhang also gave an explicit integral representation of the dual curvature measures which allowed them to define more generally for any $q \in \mathrm{R}$ the qth dual curvature measure of a convex body $K \in \mathrm{K}_o^n$ as [31, Def. 3.2]

$$\mathrm{C}_q(K,\eta) = \_1\ \rho_K(\mathbf{u})^q\, \mathrm{d}\mathrm{H}^{n-1}(\mathbf{u}). \quad (1.6)\ n \qquad \alpha_K^*(\eta)$$

Here $\alpha_K^*(\eta)$ denotes the set of directions $\mathbf{u} \in S^{n-1}$, such that the boundary point $\rho_K(\mathbf{u})\mathbf{u}$ belongs to $v_K^{-1}(\eta)$. The analog to the Minkowski–Christoffel problem in the dual Brunn– Minkowski theory is (cf. [31, Sect. 5]):

The dual Minkowski problem. Given a finite Borel measure μ on S^{n-1} and $q \in \mathrm{R}$. Find necessary and sufficient conditions for the existence of a convex body $K \in \mathrm{K}_o^n$ such that $\mathrm{C}_q(K,\cdot) = \mu$.

An amazing feature of these dual curvature measures is that they link two other well-known fundamental geometric measures of a convex body (cf. [31, Lemma 3.8]): when $q = 0$ the dual curvature measure $\mathrm{C}_0(K,\cdot)$ is – up to a factor of n – Aleksandrov's integral curvature of the polar body of K and for $q = n$ the dual curvature measure coincides with the cone-volume measure of K given by

$$\tfrac{1}{n}\, h(\mathbf{x})\,|\nabla h(\mathbf{x}) + h(\mathbf{x})\mathbf{x}|^{q-n}$$

$$\mathrm{C}_n(K,\eta) = V_K(\eta) = \_1\ \mathbf{u}, v_K(\mathbf{u})\mathrm{d}\mathrm{H}^{n-1}(\mathbf{u}).\ n \qquad v_{K^{-1}}(\eta)$$

Similarly to the Minkowski problem, solving the dual Minkowski problem is equivalent to solving a Monge–Ampère type partial differential equation if the measure μ has a density function $g : S^{n-1} \to \mathrm{R}$. In particular, the dual Minkowski problem amounts to solving the Monge–Ampère equation

$$\_^1\ h(\mathbf{x})|\nabla h(\mathbf{x}) + h(\mathbf{x})\mathbf{x}|^{q-n} \det[h_{ij}(\mathbf{x}) + \delta_{ij}h(\mathbf{x})] = g(\mathbf{x}), \quad (1.7)\ n$$

where $[h_{ij}(\mathbf{x})]$ is the Hessian matrix of the (unknown) support function h with respect to an orthonormal frame on S^{n-1}, and δ_{ij} is the Kronecker delta.

If | were omitted in (1.7), then (1.7) would become the partial differential equation of the classical Minkowski problem, see, e.g., [14,16,45]. If only the factor $|\nabla h(\mathbf{x}) + h(\mathbf{x})\,\mathbf{x}|^{q-n}$ were omitted, then equation (1.7) would become the partial differential equation associated with the cone volume measure, the so-called *logarithmic Minkowski problem* (see, e.g., [11,17]). Due to the gradient component in (1.7) the dual Minkowski problem is significantly more challenging than the classical Minkowski problem as well as the logarithmic Minkowski problem.

The cone-volume measure for convex bodies has been studied extensively over the last few years in many different contexts, see, e.g., [4,5,10–12,23,25,27–31,34,35,39–44,47,51, 55,56]. One very important property of the cone-volume measure – and which makes it so essential – is its SL(n)-invariance, or simply called affine invariance. It is also the subject of the central *logarithmic Minkowski problem* which asks for sufficient and necessary conditions of a measure μ on S^{n-1} to be the cone-volume measure $V_K(\cdot)$ of a convex body $K \in K_o^n$. This is the $p = 0$ limit case of the general *L_p-Minkowski problem* within the above mentioned L_p Brunn–Minkowski theory for which we refer to [32,38,57] and the references within.

The discrete, planar, even case of the logarithmic Minkowski problem, i.e., with respect to origin-symmetric convex polygons, was completely solved by Stancu [49,50]. Later Zhu [55] as well as Böröczky, Hegedűs and Zhu [6] settled (in particular) the case when K is a polytope of arbitrary dimension whose outer normals are in general position.

In [11], Böröczky, Lutwak, Yang and Zhang gave a complete characterization of the cone-volume measure of origin-symmetric convex bodies among even measures on the sphere. The key feature of such a measure is expressed via the following condition: A non-zero, finite Borel measure μ on the unit sphere satisfies the *subspace concentration condition* if

$$\frac{\mu(S^{n-1} \cap L)}{\mu(S^{n-1})} \le \frac{\dim L}{n} \tag{1.8}$$

for every proper subspace L of R^n, and whenever we have equality in (1.8) for some L there is a subspace L complementary to L, such that μ is concentrated on $\mathbb{S}^{n-1} \cap (L \cup L')$.

Apart from the uniqueness aspect, the symmetric case of the logarithmic Minkowski problem is settled.

Theorem 1.1 *([11]). A non-zero, finite, even Borel measure μ on S^{n-1} is the cone-volume measure of $K \in K_e^n$ if and only if μ satisfies the subspace concentration condition.*

An extension of the validity of inequality (1.8) to centered bodies, i.e., bodies whose center of mass is at the origin, was given in the discrete case by Henk and Linke [29], and in the general setting by Böröczky and Henk [7]. For a related stability result concerning (1.8) we refer to [8]. In [15], Chen, Li and Zhu proved that in the non-symmetric logarithmic Minkowski problem the subspace concentration condition is also sufficient.

A generalization (up to the equality case) of the sufficiency part of Theorem 1.1 to the qth dual curvature measure for $q \in (0, n]$ was given by Huang, Lutwak, Yang and Zhang. For clarity, we separate their main result into the next two theorems.

Theorem 1.2 *([31, Theorem 6.6]). Let $q \in (0, 1]$. A non-zero, finite, even Borel measure μ on S^{n-1} is the qth dual curvature measure of a convex body $K \in K_o^n$ if and only if μ is not concentrated on any great subsphere.*

Theorem 1.3 *([31, Theorem 6.6]). Let $q \in (1, n]$ and let μ be a non-zero, finite, even Borel measure on S^{n-1} satisfying the* subspace mass inequality

$$\frac{\mu(S^{n-1} \cap L)}{\mu(S^{n-1})} < 1 - \frac{q-1}{q}\,\frac{n-\dim L}{n-1} \tag{1.9}$$

for every proper subspace L of R^n. Then there exists an o-symmetric convex body $K \in K_e^n$ with $C_q(K, \cdot) = \mu$.

Observe that for $q = n$ the inequality (1.9) becomes essentially (1.8). In the case that the parameter q is an integer this result was strengthened by Zhao.

Theorem 1.4 *([54]). Let $q \in \{1, \dots, n-1\}$ and let μ be a non-zero, finite, even Borel measure on S^{n-1} satisfying*

$$\frac{\mu(\mathbb{S}^{n-1}\cap L)}{\mu(\mathbb{S}^{n-1})} < \min\left\{\frac{\dim L}{q}, 1\right\},$$

for every proper subspace L of $\mathbb{R}^n$. Then there exists a o-symmetric convex body $K \in \mathcal{K}_e^n$ with $C_q(K, \cdot) = \mu$.

An extension of this result to all $q \in (0, n)$ was very recently given by Böröczky, Lutwak, Yang, Zhang and Zhao [13]. That this subspace concentration bound is indeed necessary was shown by Böröczky and the authors in [9].

Theorem 1.5 *([9]). Let $K \in \mathcal{K}_e^n$, $q \in (0, n)$ and let $L \subset \mathbb{R}^n$ be a proper subspace. Then we have*

$$C_q(K, S^{n-1} \cap L) < \min\left\{\frac{\dim L}{q}, 1\right\} . C_q(K, S^{n-1})$$

The case $q < 0$ (including uniqueness) was completely settled by Zhao [53]. In particular, he proved that there is no (non-trivial) subspace concentration

Theorem 1.6 *([53]). Let $q < 0$. A non-zero, finite, even Borel measure μ on S^{n-1} is the qth dual curvature measure of a convex body $K \in \mathcal{K}_o^n$ if and only if μ is not concentrated on any closed hemisphere. Moreover, $K \in \mathcal{K}_o^n$ is uniquely determined.*

Our main result, Theorem 1.7, treats the range $q \geq n + 1$. We prove the necessity of non-trivial subspace concentration bounds on dual curvature measures of originsymmetric convex bodies.

Theorem 1.7. *Let $K \in \mathcal{K}_e^n$, $q \geq n + 1$, and let $L \subset \mathbb{R}^n$ be a proper subspace of $\mathbb{R}^n$. Then we have*

$$\frac{\widetilde{C}_q(K, \mathbb{S}^{n-1}\cap L)}{\widetilde{C}_q(K, \mathbb{S}^{n-1})} < \frac{q - n + \dim L}{q}. \tag{1.10}$$

The given upper bound is also optimal, i.e., the right hand side of (1.10) cannot be replaced by a smaller constant.

Proposition 1.8. *Let $q > n$ and $k \in \{1, ..., n - 1\}$. There exists a sequence of convex bodies $K_l \in \mathcal{K}_e^n$, $l \in \mathbb{N}$, and a k-dimensional subspace $L \subset \mathbb{R}^n$ such that*

$$\lim_{l\to\infty} \frac{\widetilde{C}_q(K_l, \mathbb{S}^{n-1}\cap L)}{\widetilde{C}_q(K_l, \mathbb{S}^{n-1})} = \frac{q-n+k}{q}.$$

In particular, the proposition above shows that the upper bound $(q - n + \dim L)/q$ would be also optimal in the range $q \in (n, n + 1)$. Unfortunately, our approach to prove (1.10) can not cover this missing range. One reason is that the following Brunn–Minkowski-type inequality (1.11) for moments of the Euclidean norm, which might be of some interest in its own, does not hold in general for $0 < p < 1$. The proof of Theorem 1.7 heavily relies on this inequality.

Theorem 1.9. *Let $p \geq 1$, and $K_0, K_1 \in \mathcal{K}^n$ with $\dim K_0 = \dim K_1 = k \geq 1$, $\mathrm{vol}_k(K_0) = \mathrm{vol}_k(K_1)$ and their affine hulls are parallel. For $\lambda \in [0, 1]$ let $K_\lambda = (1 - \lambda)K_0 + \lambda K_1$.*

Then we have

$$\int_{K_\lambda} |\boldsymbol{x}|^p \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_{1-\lambda}} |\boldsymbol{x}|^p \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \geq |2\lambda - 1|^p \left(\int_{K_0} |\boldsymbol{x}|^p \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_1} |\boldsymbol{x}|^p \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \right). \quad (1.11)$$

Equality holds if and only if

(i) $\lambda \in \{0, 1\}$, *or*

(ii) $p = 1$ *and there exists a* $\boldsymbol{u} \in S^{n-1}$ *such that* $K_0, K_1 \subset \mathrm{lin}\,\boldsymbol{u}$ *and the hyperplane* $\{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{u}, \boldsymbol{x} \rangle = 0\}$ *separates* K_λ *and* $K_{1-\lambda}$.

Here lin(·) denotes the linear hull operator. Observe for $p = 0$ the inequality also holds true by the Brunn–Minkowski inequality (cf. (2.1)). We will use the theorem in the special setting $K_0 = -K_1$ which then gives

Corollary 1.10. *Let* $K \in \mathcal{K}^n$ *with* $\dim K = k \geq 1$ *and let* $p \geq 1$. *Then for* $\lambda \in [0, 1]$ *we have*

$$\int_{(1-\lambda)K+\lambda(-K)} |\boldsymbol{x}|^p \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \geq |2\lambda - 1|^p \int_K |\boldsymbol{x}|^p \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}),$$

with equality if and only if

(i) $\lambda \in \{0, 1\}$, *or*

(ii) $p = 1$ *and there exists a* $\boldsymbol{u} \in S^{n-1}$ *such that* $K \subset \mathrm{lin}\,\boldsymbol{u}$ *and the origin is not in the relative interior of the segment* $(1 - \lambda)K + \lambda(-K)$.

In the planar case we can fill the remaining gap in the range of q, i.e., there we will prove a sharp concentration bound for all $q > 2$.

Theorem 1.11. *Let* $K \in \mathcal{K}_e^2$, $q > 2$, *and let* $L \subset \mathbb{R}^2$ *be a line through the origin. Then we have*

$$\frac{\widetilde{\mathrm{C}}_q(K, \mathbb{S}^1 \cap L)}{\mathrm{C}_q(K, \mathbb{S}^1)} < \frac{q-1}{q} \quad (1.12)$$

We remark that the logarithmic Minkowski problem as well as the dual Minkowski problem are far easier to handle for the special case where the measure μ has a positive continuous density (where subspace concentration is trivially satisfied). The singular general case for measures is substantially more delicate, which involves measure concentration and requires far more powerful techniques to solve.

The paper is organized as follows. First we will briefly recall some basic facts about convex bodies needed in our investigations in Section 2. In Section 3 we will prove Theorem 1.9, which is one of the main ingredients for the proof of the main Theorem 1.7 given in Section 4 alongside the proof of Proposition 1.8. Finally, we discuss the remaining case $q \in (n, n + 1)$ in Section 5 and prove Theorem 1.11.

2. Preliminaries

We recommend the books by Gardner [21], Gruber [26] and Schneider [48] as excellent references on convex geometry. For a given convex body $K \in \mathcal{K}^n$, the support function $h_K : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$h_K(\boldsymbol{x}) = \max_{\boldsymbol{y} \in K} \boldsymbol{x}, \boldsymbol{y}.$$

A boundary point $\boldsymbol{x} \in \partial K$ is said to have a (not necessarily unique) unit outer normal vector $\boldsymbol{u} \in \mathbb{S}^{n-1}$ if $\boldsymbol{x}, \boldsymbol{u} = h_K(\boldsymbol{u})$. The corresponding supporting hyperplane $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x}, \boldsymbol{u} = h_K(\boldsymbol{u})\}$ will be denoted by $H_K(\boldsymbol{u})$. For $K \in \mathcal{K}_o^n$, the radial function $\rho_K : \mathbb{R}^n \setminus \{\boldsymbol{0}\} \to \mathbb{R}$ is given by

$$\rho_K(\boldsymbol{x}) = \max\{\rho > 0 : \rho\boldsymbol{x} \in K\}.$$

Note, that the support function and the radial function are homogeneous of degrees 1 and −1, respectively, i.e.,

$$h_K(\lambda\boldsymbol{x}) = \lambda h_K(\boldsymbol{x}) \text{ and } \rho_K(\lambda\boldsymbol{x}) = \lambda^{-1} \rho_K(\boldsymbol{x}),$$

for $\lambda > 0$. We define the reverse radial Gauss image of $\eta \subseteq \mathbb{S}^{n-1}$ with respect to a convex

body $K \in \mathcal{K}_o^n$ by

$$\alpha_K^*(\eta) = \{\boldsymbol{u} \in \mathbb{S}^{n-1} : \rho_K(\boldsymbol{u})\boldsymbol{u} \in H_K(\boldsymbol{v}) \text{ for a } \boldsymbol{v} \in \eta\}.$$

If η is a Borel set, then $\alpha_K^*(\eta)$ is $\mathcal{H}^{n-1}$-measurable (see [48, Lemma 2.2.11]) and so the qth dual curvature measure given in (1.6) is well defined. We will need the following identity.

Lemma 2.1 *([9, Lemma 2.1]). Let $K \in \mathcal{K}_o^n$, $q > 0$ and $\eta \subseteq \mathbb{S}^{n-1}$ a Borel set. Then*

$$C_q(K,\eta) = \frac{q}{n} \int_{\{\boldsymbol{x} \in K : \boldsymbol{x}/|\boldsymbol{x}| \in \alpha_K^*(\eta)\}} |\boldsymbol{x}|^{q-n} \, d\mathcal{H}^n(\boldsymbol{x}).$$

Let L be a linear subspace of $\mathbb{R}^n$. We write $K|L$ to denote the image of the orthogonal projection of K onto L and $L^\perp$ for the subspace orthogonal to L. The linear (affine, convex) hull of a set $S \subseteq \mathbb{R}^n$ is denoted by lin$S$ (aff$S$, conv$S$), and for $\boldsymbol{v} \in \mathbb{R}^n$ we write instead of $(\text{lin}\boldsymbol{v})^\perp$ just $\boldsymbol{v}^\perp$.

As usual, for two subsets $A, B \subseteq \mathbb{R}^n$ and reals $\alpha, \beta \geq 0$, the Minkowski combination is defined by

$$\alpha A + \beta B = \{\alpha\boldsymbol{a} + \beta\boldsymbol{b} : \boldsymbol{a} \in A, \boldsymbol{b} \in B\}.$$

By the well-known Brunn–Minkowski inequality we know that the nth root of the volume of the Minkowski combination is a concave function. More precisely, for two convex bodies $K_0, K_1 \subset \mathbb{R}^n$ and for $\lambda \in [0, 1]$ we have

$$\text{vol}_n((1-\lambda)K_0 + \lambda K_1)^{1/n} \geq (1-\lambda)\text{vol}_n(K_0)^{1/n} + \lambda\text{vol}_n(K_1)^{1/n}, \qquad (2.1)$$

where $\mathrm{vol}_n(\cdot) = \mathcal{H}^n(\cdot)$ denotes the n-dimensional Hausdorff measure. We have equality in (2.1) for some $0 < \lambda < 1$ if and only if K_0 and K_1 lie in parallel hyperplanes or they are homothetic, i.e., there exist a $\boldsymbol{t} \in \mathbb{R}^n$ and $\mu \geq 0$ such that $K_1 = \boldsymbol{t} + \mu K_0$ (see, e.g., [20], [48, Sect. 6.1]).

3. A Brunn–Minkowski type inequality for moments of the Euclidean norm

In this section we will prove Theorem 1.9 and we start by recalling a variant of the well-known Karamata inequality which often appears in the context of Schur-convex functions.

Theorem 3.1 *(Karamata's inequality, see, e.g., [33, Theorem 1]). Let $D \subseteq \mathbb{R}$ be convex and let $f : D \to \mathbb{R}$ be a non-decreasing, convex function. Let $x_1, \ldots, x_k, y_1, \ldots, y_k \in D$ such that*

(i) $x_1 \geq x_2 \geq \ldots \geq x_k$,

(ii) $y_1 \geq y_2 \geq \ldots \geq y_k$,

(iii) $x_1 + x_2 + \ldots + x_i \geq y_1 + y_2 + \ldots + y_i$ *for all* $i = 1, \ldots, k$,

then

$$f(x_1) + f(x_2) + \ldots + f(x_k) \geq f(y_1) + f(y_2) + \ldots + f(y_k). \qquad (3.1)$$

If f is strictly convex, then equality in (3.1) holds if and only if $x_i = y_i$, $i = 1, \ldots, k$.

As a consequence we obtain an estimate for the value of powers of convex combinations of real numbers.

Lemma 3.2. *Let $p \geq 1$, $z, \bar{z} \in \mathbb{R}$ and $\lambda \in [0, 1]$. Then*

$$|\lambda z + (1-\lambda)\bar{z}|^p + |\lambda \bar{z} + (1-\lambda)z|^p \geq |2\lambda - 1|^p(|z|^p + |\bar{z}|^p)$$

and equality holds if and only if at least one of the following statements is true:

(i) $\lambda \in \{0, 1\}$,

(ii) $\bar{z} = -z$,

(iii) $p = 1$, $z\bar{z} < 0$ *and* $\max\{\lambda, 1-\lambda\} \geq \frac{\max\{|z|,|\bar{z}|\}}{|z|+|\bar{z}|}$.

Proof. By symmetry we may assume $\lambda \geq \frac{1}{2}$, $z \geq |\bar{z}|$ and $z \geq 0$. Write

$$x_1 = |\lambda z + (1-\lambda)\bar{z}|, \; x_2 = |\lambda \bar{z} + (1-\lambda)z|, \; y_1 = (2\lambda - 1)z, \; y_2 = (2\lambda - 1)|\bar{z}|.$$

We want to apply Karamata's inequality with $D = \mathbb{R}_{\geq 0}$ and $f(t) = t^p$. We readily have

$$y_1 \geq y_2,$$

$$x_1^2 - x_2^2 = (2\lambda - 1)(z^2 - \bar{z}^2) \geq 0,$$

and since $\bar{z} \geq -z$ we also have

$$x_1 \geq y_1.$$

It remains to show that $x_1 + x_2 \geq y_1 + y_2$. The triangle inequality gives

$$|(\lambda z + (1-\lambda)\bar{z}) \pm (\lambda \bar{z} + (1-\lambda)z)| \leq x_1 + x_2.$$

Hence

$$x_1 + x_2 \geq \max\{(2\lambda - 1)|z - \bar{z}|, |z + \bar{z}|\}$$

$$\geq (2\lambda - 1)\max\{|z - \bar{z}|, |z + \bar{z}|\}$$

$$= (2\lambda - 1)(z + |\bar{z}|) = y_1 + y_2$$

and Karamata's inequality (3.1) yields $x_1^p + x_2^p \geq y_1^p + y_2^p$, i.e., the inequality of the lemma.

Suppose now we have equality and as before we assume $\lambda \geq \frac{1}{2}$, $|z| \geq |\bar{z}|$ and $z \geq 0$. Let $\lambda < 1$ and first let $p > 1$. In this case the equality condition of Karamata's inequality asserts $x_1 = y_1$ and since $\bar{z} \geq -z$, $\lambda \geq \frac{1}{2}$ we conclude $\bar{z} = -z$.

Now suppose $p = 1$, $\lambda < 1$ and $z = -\bar{z}$. In view of our assumptions we have $\lambda z + (1-\lambda)\bar{z} \geq 0$ and so if

$$(2\lambda - 1)(z + |\bar{z}|) = y_1 + y_2 = x_1 + x_2 = \lambda z + (1-\lambda)\bar{z} + |\lambda \bar{z} + (1-\lambda)z|$$

then $\lambda \bar{z} + (1-\lambda)z \leq 0$ and $\bar{z} < 0$. Thus

$$z\lambda \geq \frac{-\bar{z}}{z - \bar{z}}.$$

On the other hand, condition (iii) implies $x_1 + x_2 = y_1 + y_2$. □

The next lemma allows us to replace spheres appearing as level sets of the norm function by hyperplanes. It appeared first in Alesker [1] and for more explicit versions of it we refer to [46, Lemma 2.1] and [3, (10.4.2)].

Lemma 3.3. *Let $p \geq 1$. There is a constant $c = c(n, p)$ such that for every $\boldsymbol{x} \in \mathbb{R}^n$*

$$|\boldsymbol{x}|^p = c \cdot \int_{\mathbb{S}^{n-1}} |\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p \, d\mathcal{H}^{n-1}(\boldsymbol{\vartheta}).$$

Now we come to the proof of Theorem 1.9. Based on Lemma 3.3 it will be sufficient to prove (1.11) if the integrand $|\boldsymbol{x}|^p$ is replaced by $|\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p$ for an arbitrary but fixed $\boldsymbol{\vartheta} \in \mathbb{S}^{n-1}$. This will be done in the next Lemma 3.4, and the final proof of Theorem 1.9 together with the classification of the equality case will be given afterwards.

Lemma 3.4. *Let $p \geq 1$, and $K_0, K_1 \in \mathcal{K}^n$ with* $\dim K_0 = \dim K_1 = k \geq 1$, $\mathrm{vol}_k(K_0) = \mathrm{vol}_k(K_1)$ *and their affine hulls are parallel. Let $\boldsymbol{\vartheta} \in \mathbb{S}^{n-1}$ and for $\lambda \in [0, 1]$ let $K_\lambda = (1-\lambda)K_0 + \lambda K_1$. Then*

$$\int_{K_\lambda} |\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p \, d\mathcal{H}^k(\boldsymbol{x}) + \int_{K_{1-\lambda}} |\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p \, d\mathcal{H}^k(\boldsymbol{x})$$

$$\geq |2\lambda - 1|^p \left(\int_{K_0} |\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_1} |\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \right). \qquad (3.2)$$

Proof. The proof basically follows the Kneser–Süss proof of the Brunn–Minkowski inequality as given in [48, Proof of Theorem 7.1.1], and then at the end we use Lemma 3.2.

Without loss of generality we assume that $\mathrm{vol}_k(K_0) = \mathrm{vol}(K_1) = 1$. The Brunn– Minkowski-inequality (2.1) gives in this setting for any $\lambda \in [0, 1]$

$$\mathrm{vol}_k(K_\lambda) \geq 1. \qquad (3.3)$$

For $\alpha \in \mathbb{R}$ denote

$$H(\boldsymbol{\theta}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{x}, \boldsymbol{\theta} \rangle = \alpha\}, \quad H^-(\boldsymbol{\theta}, \alpha) = \{\boldsymbol{x} \in \mathbb{R}^n : \langle \boldsymbol{x}, \boldsymbol{\theta} \rangle \leq \alpha\}.$$

First suppose that K_0 lies in a hyperplane parallel to $\boldsymbol{\vartheta}^\perp$(and therefore K_λ for $\lambda \in [0, 1]$), i.e., $K_i \subset H(\boldsymbol{\vartheta}, \alpha_i)$, $\alpha_i \in \mathbb{R}$, $i \in \{0, 1\}$. By (3.3) and Lemma 3.2 we find

$$\begin{aligned}
&\int_{K_\lambda} |\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_{1-\lambda}} |\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \\
&= \mathrm{vol}_k(K_\lambda)|(1-\lambda)\alpha_0 + \lambda\alpha_1|^p + \mathrm{vol}_k(K_{1-\lambda})|\lambda\alpha_0 + (1-\lambda)\alpha_1|^p \\
&\geq |(1-\lambda)\alpha_0 + \lambda\alpha_1|^p + |\lambda\alpha_0 + (1-\lambda)\alpha_1|^p \\
&\geq |2\lambda - 1|^p(|\alpha_0|^p + |\alpha_1|^p) \\
&= |2\lambda - 1|^p(\mathrm{vol}_k(K_0)|\alpha_0|^p + \mathrm{vol}_k(K_1)|\alpha_1|^p) \\
&= |2\lambda - 1|^p \left(\int_{K_0} |\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_1} |\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p \, \mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \right)
\end{aligned}$$

Thus, in the following we may assume that $K_0 \not\subseteq \boldsymbol{v} + \boldsymbol{\theta}^\perp$ for all $\boldsymbol{v} \in \mathbb{R}^n$ and hence, $-h_{K_i}(-\boldsymbol{\vartheta}) < h_{K_i}(\boldsymbol{\vartheta})$ for $i = 0, 1$. For $t \in \mathbb{R}$ and for $i = 0, 1$ we set

$$v_i(t) = \mathrm{vol}_{k-1}(K_i \cap H(\boldsymbol{\vartheta}, t)), \quad w_i(t) = \mathrm{vol}_k(K_i \cap H^-(\boldsymbol{\vartheta}, t)),$$

so that $w_i(t) = \int_{-\infty}^{t} v_i(\zeta)\, \mathrm{d}\zeta$, $i \in \{0, 1\}$. On $(-h_{K_i}(-\boldsymbol{\vartheta}), h_{K_i}(\boldsymbol{\vartheta}))$ the function v_i is continuous and hence w_i is differentiable. For $i \in \{0, 1\}$ let z_i be the inverse function of w_i. Then z_i is differentiable with

$$z_i'(\tau) = \frac{1}{w_i'(z_i(\tau))} = \frac{1}{v_i(z_i(\tau))} \qquad (3.4)$$

for $\tau \in (0, 1)$. Writing $z_\mu(\tau) = (1-\mu)z_0(\tau) + \mu z_1(\tau)$ for $\mu \in [0, 1]$ we have

$$\int_{K_\mu} |\langle \boldsymbol{x}, \boldsymbol{\vartheta}\rangle|^p \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) = \int_{\mathbb{R}} |t|^p \mathrm{vol}_{k-1}(K_\mu \cap H(\boldsymbol{\vartheta}, t))\,\mathrm{d}t$$

$$= \int_0^1 |z_\mu(\tau)|^p \mathrm{vol}_{k-1}(K_\mu \cap H(\boldsymbol{\theta}, z_\mu(\tau))) z'_\mu(\tau)\,\mathrm{d}\tau. \qquad (3.5)$$

Since $K_\mu \cap H(\boldsymbol{\vartheta}, z_\mu(\tau)) \supseteq (1-\mu)(K_0 \cap H(\boldsymbol{\vartheta}, z_0(\tau))) + \mu(K_1 \cap H(\boldsymbol{\vartheta}, z_1(\tau)))$ we may apply the Brunn–Minkowski inequality to the latter set and together with (3.4) we get

$$\mathrm{vol}_{k-1}(K_\mu \cap H(\boldsymbol{\theta}, z_\mu(\tau))) z'_\mu(\tau) \geq \mathrm{vol}_{k-1}\left((1-\mu)(K_0 \cap H(\boldsymbol{\theta}, z_0(\tau))) + \mu(K_1 \cap H(\boldsymbol{\theta}, z_1(\tau)))\right) z'_\mu(\tau)$$

$$\geq \left((1-\mu) v_0(z_0(\tau))^{\frac{1}{k-1}} + \mu v_1(z_1(\tau))^{\frac{1}{k-1}}\right)^{k-1} \left(\frac{1-\mu}{v_0(z_0(\tau))} + \frac{\mu}{v_1(z_1(\tau))}\right) \qquad (3.6)$$

$$\geq \left(v_0(z_0(\tau))^{\frac{1-\mu}{k-1}} v_1(z_1(\tau))^{\frac{\mu}{k-1}}\right)^{k-1} v_0(z_0(\tau))^{-(1-\mu)} v_1(z_1(\tau))^{-\mu}$$

$$= 1,$$

where for the last inequality we used the weighted arithmetic/geometric-mean inequality.
Therefore, (3.5) and (3.6) yield

$$\int_{K_\lambda} |\langle \boldsymbol{x}, \boldsymbol{\vartheta}\rangle|^p \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_{1-\lambda}} |\langle \boldsymbol{x}, \boldsymbol{\vartheta}\rangle|^p \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x})$$

$$\geq \int_0^1 |z_\lambda(\tau)|^p + |z_{1-\lambda}(\tau)|^p \,\mathrm{d}\tau$$

$$= \int_0^1 |\lambda z_1(\tau) + (1-\lambda) z_0(\tau)|^p + |\lambda z_0(\tau) + (1-\lambda) z_1(\tau)|^p \,\mathrm{d}\tau.$$

Next in order to estimate the integrand we use Lemma 3.2 and then we substitute back via (3.4)

$$\int_{K_\lambda} |\langle \boldsymbol{x}, \boldsymbol{\vartheta}\rangle|^p \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_{1-\lambda}} |\langle \boldsymbol{x}, \boldsymbol{\vartheta}\rangle|^p \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x})$$

$$\geq \int_0^1 |\lambda z_1(\tau) + (1-\lambda) z_0(\tau)|^p + |\lambda z_0(\tau) + (1-\lambda) z_1(\tau)|^p \,\mathrm{d}\tau$$

$$\geq \int_0^1 |2\lambda - 1|^p \left(|z_0(\tau)|^p + |z_1(\tau)|^p\right) \mathrm{d}\tau \qquad (3.7)$$

$$= |2\lambda - 1|^p \left(\int_0^1 |z_0(\tau)|^p \,\mathrm{d}\tau + \int_0^1 |z_1(\tau)|^p \,\mathrm{d}\tau\right)$$

$$= |2\lambda - 1|^p \left(\int_0^1 |z_0(\tau)|^p v_0(z_0(\tau)) \cdot z_0'(\tau)\,\mathrm{d}\tau + \int_0^1 |z_1(\tau)|^p v_1(z_1(\tau)) \cdot z_1'(\tau)\,\mathrm{d}\tau \right)$$

$$= |2\lambda - 1|^p \left(\int_{\mathbb{R}} |t|^p \mathrm{vol}_{k-1}(K_0 \cap H(\boldsymbol{\vartheta}, t))\,\mathrm{d}t + \int_{\mathbb{R}} |t|^p \mathrm{vol}_{k-1}(K_1 \cap H(\boldsymbol{\vartheta}, t))\,\mathrm{d}t \right)$$

$$= |2\lambda - 1|^p \left(\int_{K_0} |\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_1} |\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \right)$$

Now we are ready to prove Theorem 1.9.

Proof of Theorem 1.9. Without loss of generality we assume that $\mathrm{vol}_k(K_0) = \mathrm{vol}(K_1) = 1$.

In order to prove the desired inequality (1.11) we first substitute there the integrand $|\boldsymbol{x}|^p$ via Lemma 3.3 which leads, after an application of Fubini, to the equivalent inequality

$$\int_{S^{n-1}} \left(\int_{K_\lambda} |\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_{1-\lambda}} |\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \right) \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{\vartheta}) \geq |2\lambda - 1|^p \times \tag{3.8}$$

$$\int_{S^{n-1}} \left(\int_{K_0} |\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) + \int_{K_1} |\langle \boldsymbol{x}, \boldsymbol{\vartheta} \rangle|^p \,\mathrm{d}\mathcal{H}^k(\boldsymbol{x}) \right) \mathrm{d}\mathcal{H}^{n-1}(\boldsymbol{\vartheta}).$$

By Lemma 3.4 this inequality holds pointwise for every $\boldsymbol{\vartheta} \in S^{n-1}$. Hence we have shown (3.8) and thus (1.11).

Now suppose that equality holds in (1.11) for two k-dimensional convex bodies K_0, K_1 of k-dimensional volume 1. We may assume $\lambda \in (0, 1)$. Then we also have equality in

(3.2) for any $\boldsymbol{\vartheta} \in S^{n-1}$ and we choose a $\boldsymbol{\vartheta} \in S^{n-1}$ such that $K_0 \nsubseteq \boldsymbol{v} + \boldsymbol{\theta}^{\perp}$ for any $\boldsymbol{v} \in \mathbb{R}^n$.

We use the notation $z_\lambda(\cdot)$ as given in the proof of Lemma 3.4.

Then the equality in (3.2) implies equality in (3.6) and thus

$$\mathrm{vol}_k(K_\lambda) = \int_0^1 \mathrm{vol}_{k-1}(K_\lambda \cap H(\boldsymbol{\theta}, z_\lambda(\tau)))z'_\lambda(\tau)\,\mathrm{d}\tau$$

$$= 1 = \lambda \mathrm{vol}_k(K_0) + (1-\lambda)\mathrm{vol}_k(K_1).$$

Hence, by the equality conditions of the Brunn–Minkowski inequality, K_0 and K_1 are homothets and we conclude $K_0 = \boldsymbol{v} + K_1$ for some $\boldsymbol{v} \in \mathbb{R}^n$. Thus for $\tau \in [0, 1]$

$$z_0(\tau) = z_1(\tau) + \langle\boldsymbol{\vartheta}, \boldsymbol{v}\rangle. \tag{3.9}$$

Since we must also have equality in (3.7) the equality conditions (ii), (iii) of Lemma 3.2 can be applied to $z_i(\tau)$, $i = 0, 1$. If $p > 1$ then Lemma 3.2 (ii) implies $z_0(\tau) = -z_1(\tau)$ for $\tau \in [0, 1]$. Together with (3.9), however, we get the contradiction that $z_0(\tau) = \frac{1}{2}\langle\boldsymbol{\vartheta}, \boldsymbol{v}\rangle$ is constant.

Thus we must have $p = 1$ and in this case we get from Lemma 3.2 (iii) and (3.9)

$$0 \geq z_0(\tau)z_1(\tau) = z_0(\tau)(z_0(\tau) - \langle\boldsymbol{\theta}, \boldsymbol{v}\rangle), \tag{3.10}$$

$$\max\{\lambda, 1-\lambda\} \geq \frac{\max\{|z_0(\tau)|, |z_1(\tau)|\}}{|z_0(\tau)| + |z_1(\tau)|}. \tag{3.11}$$

For a sufficiently small positive ε the right hand side of (3.10), as a function in z_0, is positive on some interval $(-\varepsilon, 0)$ or $(0, \varepsilon)$. If

$$h_{K_0}(-\boldsymbol{\vartheta}), h_{K_0}(\boldsymbol{\vartheta}) > 0 \tag{3.12}$$

then the domain $[-h_{K_0}(-\boldsymbol{\vartheta}), h_{K_0}(\boldsymbol{\vartheta})]$ of $z_0(\tau)$ contains a sufficiently small neighborhood of 0 which then would contradict (3.10).

Hence we can assume that there exists no $\boldsymbol{\vartheta} \in S^{n-1}$ satisfying (3.12) which means that there exists no two points in K_0 which can be strictly separated by a hyperplane containing $\mathbf{0}$. Thus we can assume that K_0 is a segment contained in a line $\mathrm{lin}\,\boldsymbol{u}$ for some $\boldsymbol{u} \in S^{n-1}$ and the origin is not a relative interior point of K_0. Interchanging the roles of K_0 and K_1 in the argumentation above leads to the same conclusion for K_1 and since the affine hulls of K_0 and K_1 are parallel we also have $K_1 \subset \mathrm{lin}\,\boldsymbol{u}$.

So without loss of generality let $K_0 = [\alpha_0, \alpha_0+1]\cdot\boldsymbol{u}$, $\alpha_0 \geq 0$, and $K_1 = [\alpha_1, \alpha_1+1]\cdot\xi\boldsymbol{u}$ with $\alpha_1 \geq 0$ and $\xi \in \{\pm 1\}$. We may choose $\boldsymbol{\vartheta} = \boldsymbol{u}$ and the inequality $0 \geq z_0(\tau)z_1(\tau)$ (cf. (3.10)) gives $\xi = -1$, i.e., $K_1 = [-\alpha_1 - 1, -\alpha_1]\cdot\boldsymbol{u}$. It remains to show that equality in (1.11) in this setting is equivalent to $\boldsymbol{u}^\perp$ separating K_λ and $K_{1-\lambda}$.

By symmetry we may assume $\lambda \geq \frac{1}{2}$ and by the choice of $\boldsymbol{\vartheta}$ we get

$$z_0(\tau) = \alpha_0 + \tau,\ z_1(\tau) = -(\alpha_1 + 1 - \tau).$$

The equality condition (3.11) yields

$$\lambda \geq \frac{\max\{\alpha_0+\tau, \alpha_1+1-\tau\}}{\alpha_0+\alpha_1+1}$$

for all $\tau \in [0, 1]$ and so

$$\lambda \geq \frac{\max\{\alpha_0, \alpha_1\}+1}{\alpha_0+\alpha_1+1}.$$

Therefore $(1-\lambda)(\alpha_0+1)-\lambda\alpha_1 \leq 0$ and $\lambda\alpha_0-(1-\lambda)(\alpha_1+1) \geq 0$, which in turn gives

$$K_\lambda = [(1-\lambda)\alpha_0-\lambda(\alpha_1+1), (1-\lambda)(\alpha_0+1)-\lambda\alpha_1]\cdot \boldsymbol{u} \subset H^-(\boldsymbol{u},0),$$
$$K_{1-\lambda} = [\lambda\alpha_0-(1-\lambda)(\alpha_1+1), \lambda(\alpha_0+1)-(1-\lambda)\alpha_1]\cdot \boldsymbol{u} \subset H^-(-\boldsymbol{u},0). \tag{3.13}$$

Thus $\boldsymbol{u}^\perp$ separates K_λ and $K_{1-\lambda}$. It remains to show that this separating property also implies equality in (1.11). If there exists such an $\boldsymbol{u}$ then we may assume that K_λ and $K_{1-\lambda}$ are given as in (3.13) and so

$$\int_{K_\lambda}|\boldsymbol{x}|\,\mathrm{d}H(\boldsymbol{x}) + \int_{K_{1-\lambda}}|\boldsymbol{x}|\,\mathrm{d}H(\boldsymbol{x})$$
$$= \int_{(1-\lambda)\alpha_0-\lambda(\alpha_1+1)}^{(1-\lambda)(\alpha_0+1)-\lambda\alpha_1} -t\,\mathrm{d}t + \int_{\lambda\alpha_0-(1-\lambda)(\alpha_1+1)}^{\lambda(\alpha_0+1)-(1-\lambda)\alpha_1} t\,\mathrm{d}t$$
$$= \frac{1}{2}\Big((\lambda(\alpha_0+1)-(1-\lambda)\alpha_1)^2-(\lambda\alpha_0-(1-\lambda)(\alpha_1+1))^2$$
$$-((1-\lambda)(\alpha_0+1)-\lambda\alpha_1)^2+((1-\lambda)\alpha_0-\lambda(\alpha_1+1))^2\Big)$$
$$= (2\lambda-1)(\alpha_0+\alpha_1+1)$$
$$= (2\lambda-1)\cdot\frac{1}{2}\left((\alpha_0+1)^2-\alpha_0^2+(\alpha_1+1)^2-\alpha_1^2\right)$$
$$= (2\lambda-1)\left(\int_{K_0}|\boldsymbol{x}|\,\mathrm{d}H(\boldsymbol{x}) + \int_{K_1}|\boldsymbol{x}|\,\mathrm{d}H(\boldsymbol{x})\right). \qquad \square$$

We remark that the factor $|2\lambda-1|^\rho$ in Theorem 1.9 (as well as in Corollary 1.10) cannot be replaced by a smaller one. This can be seen from the following example. Let $C \in \mathcal{K}_e^n$, $\boldsymbol{u} \in \mathbb{S}^{n-1}$, $\rho \in \mathbb{R}_{>0}$, $K_0 = C + \rho\cdot\boldsymbol{u}$ and $K_1 = -K_0$. Then, for every $\mu \in [0, 1]$ we have

$|\boldsymbol{x}|^p \, d\mathcal{H}^n(\boldsymbol{x}) = \rho^p \; |\rho^{-1}\boldsymbol{x} - (2\mu - 1)\boldsymbol{u}|^p \, d\mathcal{H}^n(\boldsymbol{x}).$

K_μ C

Moreover, the symmetry of C gives $K_{1-\mu} = -K_\mu$ and therefore, for a given $\lambda \in [0, 1]$, we conclude

$K_{1-\lambda}$ $|\boldsymbol{x}|^p \, d\mathcal{H}^n(\boldsymbol{x}) = \lim$ $K_{\lambda 0}$ $|\,|\boldsymbol{x}|\,|^p \, d\mathcal{H}^n(\boldsymbol{x}) = |2\lambda - 1|^p.$

$\rho\to\infty$ K^1 $|\boldsymbol{x}|^p \, d\mathcal{H}^n(\boldsymbol{x})$ $\rho\to\infty$ K $\boldsymbol{x}^p$ d $^n(\boldsymbol{x})$ lim

4. Proof of Theorem 1.7

Now we are ready to prove Theorem 1.7. We use Fubini's theorem to decompose the dual curvature measure of $K \in \mathcal{K}_e^n$ into integrals over sections with affine planes orthogonal to the given subspace L. In order to compare these integrals with the corresponding integrals over sections associated to the dual curvature measure of the Borel set $S^{n-1} \cap L$, we apply Corollary 1.10.

Proof of Theorem 1.7. Let $\dim L = k \in \{1, \dots, n-1\}$, and for $\boldsymbol{y} \in K|L$ let $\overline{\boldsymbol{y}} = \rho_{K|L}(\boldsymbol{y})\boldsymbol{y} \in \partial(K|L)$,

$$K_{\boldsymbol{y}} = K \cap (\boldsymbol{y} + L^\perp),$$

$$M_{\boldsymbol{y}} = \mathrm{conv}\{K_{\boldsymbol{0}}, K_{\overline{\boldsymbol{y}}}\}.$$

By Lemma 2.1, Fubini's theorem and the fact that $M_{\boldsymbol{y}} \cap (\boldsymbol{y} + L^\perp) \subseteq K_{\boldsymbol{y}}$ we may write

$$\tilde{C}_q(K, S^{n-1}) = \frac{q}{n} \int_{K|L} \left(\int_{K_{\boldsymbol{y}}} |\boldsymbol{z}|^{q-n} \, d\mathcal{H}^{n-k}(\boldsymbol{z}) \right) d\mathcal{H}^k(\boldsymbol{y}) \tag{4.1}$$

$$\geq \frac{q}{n} \int_{K|L} \left(\int_{M_{\boldsymbol{y}} \cap (\boldsymbol{y}+L^\perp)} |\boldsymbol{z}|^{q-n} \, d\mathcal{H}^{n-k}(\boldsymbol{z}) \right) d\mathcal{H}^k(\boldsymbol{y}).$$

In order to estimate the inner integral we fix a $\boldsymbol{y} \in K|L$, $\boldsymbol{y} \neq \boldsymbol{0}$, and for abbreviation we set $\tau = \rho_{K|L}(\boldsymbol{y})^{-1} \leq 1$. Then, by the symmetry of K we find

$$M_{\boldsymbol{y}} \cap (\boldsymbol{y} + L^\perp) \supseteq \tau(K_{\overline{\boldsymbol{y}}}) + (1-\tau)(K_{\boldsymbol{0}})$$

$$\supseteq \tau(K_{\overline{\boldsymbol{y}}}) + (1-\tau)\left(\frac{1}{2}K_{\overline{\boldsymbol{y}}} + \frac{1}{2}(-K_{\overline{\boldsymbol{y}}})\right) = \frac{1+\tau}{2} K_{\overline{\boldsymbol{y}}} + \frac{1-\tau}{2}(-K_{\overline{\boldsymbol{y}}}).$$

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Hence, $M_{\boldsymbol{y}} \cap (\boldsymbol{y} + L^{\perp})$ contains a convex combination of a set and its reflection at the origin. This allows us to apply Corollary 1.10 from which we obtain

$$\int_{M_{\boldsymbol{y}} \cap (\boldsymbol{y}+L^{\perp})} |\boldsymbol{z}|^{q-n}\, d\mathcal{H}^{n-k}(\boldsymbol{z}) \geq \int_{\frac{1+\tau}{2} K_{\boldsymbol{y}} + \frac{1-\tau}{2}(-K_{\boldsymbol{y}})} |\boldsymbol{z}|^{q-n}\, d\mathcal{H}^{n-k}(\boldsymbol{z})$$

$$\geq \tau^{q-n} \int_{K_{\boldsymbol{y}}} |\boldsymbol{z}|^{q-n}\, d\mathcal{H}^{n-k}(\boldsymbol{z})$$

for every $\boldsymbol{y} \in K|L$, $\boldsymbol{y} \neq \boldsymbol{0}$. By the equality characterization in Corollary 1.10 the last inequality is strict whenever $\tau < 1$, i.e., $\boldsymbol{y}$ belongs to the relative interior of $K|L$. Together with (4.1) we obtain the lower bound

$$C_q(K, \mathbb{S}^{n-1}) > \frac{q}{n} \int_{K|L} \left(\rho_{K|L}(\boldsymbol{y})^{n-q} \int_{K_{\boldsymbol{y}}} |\boldsymbol{z}|^{q-n}\, d\mathcal{H}^{n-k}(\boldsymbol{z}) \right) d\mathcal{H}^{k}(\boldsymbol{y}). \quad (4.2)$$

In order to evaluate $C_q(K, \mathbb{S}^{n-1} \cap L)$ we note that for $\boldsymbol{x} \in K$ we have $\boldsymbol{x}/|\boldsymbol{x}| \in \alpha_K^*(\mathbb{S}^{n-1} \cap L)$ if and only if the boundary point $\rho_K(\boldsymbol{x})\boldsymbol{x}$ has an outer unit normal in L. Hence,

$$\{\boldsymbol{0}\} \cup \{\boldsymbol{x} \in K : \boldsymbol{x}/|\boldsymbol{x}| \in \alpha_K^*(\mathbb{S}^{n-1} \cap L)\} = \bigcup_{\boldsymbol{v} \in \partial(K|L)} \operatorname{conv}\{\boldsymbol{0}, K_{\boldsymbol{v}}\},$$

and in view of Lemma 2.1 and Fubini's theorem we may write

$$C_q(K, \mathbb{S}^{n-1} \cap L) = \frac{q}{n} \int_{K|L} \left(\int_{\operatorname{conv}\{\boldsymbol{0}, K_{\boldsymbol{y}}\} \cap (\boldsymbol{y}+L^{\perp})} |\boldsymbol{z}|^{q-n}\, d\mathcal{H}^{n-k}(\boldsymbol{z}) \right) d\mathcal{H}^{k}(\boldsymbol{y})$$

$$= \frac{q}{n} \int_{K|L} \left(\int_{\rho_{K|L}(\boldsymbol{y})^{-1} K_{\boldsymbol{y}}} |\boldsymbol{z}|^{q-n}\, d\mathcal{H}^{n-k}(\boldsymbol{z}) \right) d\mathcal{H}^{k}(\boldsymbol{y})$$

$$= \frac{q}{n} \int_{K|L} \rho_{K|L}(\boldsymbol{y})^{k-q} \left(\int_{K_{\boldsymbol{y}}} |\boldsymbol{z}|^{q-n} \, \mathrm{d}\mathcal{H}^{n-k}(\boldsymbol{z}) \right) \mathrm{d}\mathcal{H}^{k}(\boldsymbol{y}).$$

The inner integral is independent of the length of $\boldsymbol{y} \in K|L$ and might be as well considered as the value $g(\boldsymbol{u})$ of a (measurable) function $g \colon S^{n-1} \cap L \to \mathbb{R}_{\geq 0}$, i.e.,

$$g(\boldsymbol{u}) = \int_{K_{\boldsymbol{u}}} |\boldsymbol{z}|^{q-n} \, \mathrm{d}\mathcal{H}^{n-k}(\boldsymbol{z}).$$

With this notation and using spherical coordinates we obtain

$$C_q(K, S^{n-1} \cap L)$$

$$= \frac{q}{n} \int_{K|L} \rho_{K|L}(\boldsymbol{y})^{k-q} g(\boldsymbol{y}/|\boldsymbol{y}|) \, \mathrm{d}\mathcal{H}^{k}(\boldsymbol{y})$$

$$= \frac{q}{n} \int_{S^{n-1} \cap L} \int_0^{\rho_{K|L}(\boldsymbol{u})} g(\boldsymbol{u}) \rho_{K|L}(r\boldsymbol{u})^{k-q} r^{k-1} \, \mathrm{d}r \, \mathrm{d}\mathcal{H}^{k-1}(\boldsymbol{u})$$

$$= \frac{q}{n} \int_{S^{n-1} \cap L} g(\boldsymbol{u}) \rho_{K|L}(\boldsymbol{u})^{k-q} \left(\int_0^{\rho_{K|L}(\boldsymbol{u})} r^{q-1} \, \mathrm{d}r \right) \mathrm{d}\mathcal{H}^{k-1}(\boldsymbol{u}) \tag{4.3}$$

$$= \frac{1}{n} \int_{S^{n-1} \cap L} g(\boldsymbol{u}) \rho_{K|L}(\boldsymbol{u})^{k} \, \mathrm{d}\mathcal{H}^{k-1}(\boldsymbol{u}).$$

Applying the same transformation to the right hand side of (4.2) leads to

$$C_q(K, S^{n-1}) >$$

$$\frac{q}{n} \int_{K|L} \rho_{K|L}(\boldsymbol{y})^{n-q} g(\boldsymbol{y}/|\boldsymbol{y}|) \, \mathrm{d}\mathcal{H}^{k}(\boldsymbol{y})$$

$$= \frac{q}{n} \int_{S^{n-1} \cap L} g(\boldsymbol{u}) \left(\int_0^{\rho_{K|L}(\boldsymbol{u})} \rho_{K|L}(r\boldsymbol{u})^{n-q} r^{k-1} \, \mathrm{d}r \right) \mathrm{d}\mathcal{H}^{k-1}(\boldsymbol{u}) \tag{4.4}$$

$$ \_ \quad \Big(\rho_{K|L}(u) \quad \Big)_H $$

$$ = \frac{q}{n} \int_{S^{n-1}\cap L} g(u)\rho_{K|L}(u)^{n-q} \left(\int_0 r^{q-n+k-1}\,\mathrm{d}r \right) \mathrm{d}\mathcal{H}^{k-1}(u) $$

$$ = \frac{q}{n}\,\frac{1}{q-n+k} \int_{S^{n-1}\cap L} g(u)\rho_{K|L}(u)^k \,\mathrm{d}\mathcal{H}^{k-1}(u). $$

Combining (4.3) and (4.4) gives the desired bound

$$ \frac{C_q(K, S^{n-1}\cap L)}{C_q(K, S^{n-1})} < \frac{q-n+k}{q}. $$

Next we show that the bounds given in Theorem 1.7 are tight for every choice of $q \geq n+1$.

Proof of Proposition 1.8. Let $k \in \{1, ..., n-1\}$ and for $l \in \mathbb{N}$, let K_l be the cylinder given as the cartesian product of two lower-dimensional balls

$$ K_l = (lB_k) \times B_{n-k}. $$

Let $L = \mathrm{lin}\{e_1, ..., e_k\}$ be the k-dimensional subspace generated by the first k canonical unit vectors e_i. For $x \in \mathbb{R}^n$ write $x = x_1 + x_2$, where $x_1 = x|L$ and $x_2 = x|L^\perp$. The supporting hyperplane of K_l with respect to a unit vector $v \in S^{n-1} \cap L$ is given by

$$ H_{K_l}(v) = \{x \in \mathbb{R}^n : \langle v, x_1\rangle = l\}. $$

Hence the part of the boundary of K_l covered by all these supporting hyperplanes is given by $lS^{k-1} \times B_{n-k}$. In view of Lemma 2.1 and Fubini's theorem we conclude

$$ C_q(K_l, S^{n-1}\cap L) = \frac{q}{2n} \int_{l B_k} \left(\int_{\{x_2 \in B_{n-k}:\, l|x_2| \leq |x_1|\}} (|x_1|^2 + |x_2|^2)^{\frac{q-n}{2}} \,\mathrm{d}\mathcal{H}^{n-k}(x_2) \right) \mathrm{d}\mathcal{H}^k(x_1). \qquad (4.5) $$

Denote the volume of B_n by ω_n. Recall, that the surface area of B_n is given by $n\omega_n$ and for abbreviation we set

$$ c = c(q,k,n) = \frac{q}{n} k\omega_k (n-k)\omega_{n-k}. $$

Switching to the cylindrical coordinates

$$ x_1 = su, \qquad s \geq 0, u \in S^{k-1}, \qquad x_2 = tv, \qquad t \geq 0, v \in S^{n-k-1}, $$

transforms the right hand side of (4.5) to

$$ C_q(K_l, S_{n-1}\cap L) = \int^{l} \int^{s/l} c s^{k-1} t^{n-k-1} (s^2 + t^2)^{\frac{q-n}{2}} \,\mathrm{d}t\,\mathrm{d}s $$

0 0

$$= c \int_0^l \int_0^1 s^{q-1} l^{-1} t^{k-1} \ldots (1 + l^{-2} t^2)^{\frac{q-n}{2}} \, 1\, k\, n\, n\, k\, 1 \quad 2\, 2\, q-n \, \mathrm{d}t\mathrm{d}s$$

$$= cl^q \quad n+k \quad 1\ 1 \quad q\ 1\ n\ k\ 1 \quad 2\ 2\ q- \quad - \int_0^1 \int_0^1 s^{-} t^{--} (1 + l^{-} t\,)^{-\frac{n}{2}} \mathrm{d}t\mathrm{d}s.$$

Analogously we obtain

$$C_q(K_l, S_{n-1}) = \_q\ {}_{1\ k} \Big(\Big| \Big\backslash_{2\,n} \ (|x_1|^2 + |x_2|^2)^{\frac{q-2n}{}} \mathrm{d}H_{n-k}(x_2) \Big) \Big| \mathrm{d}H_k(x_1)\ n$$

$x \in lB\ x \in B\ {}_{-k}\ k\ 1\ n\ k\ 1\ 2\ 2\ q-n$

$$= \int_0^l \int_0^1 \ {}^{-} t^{--} (s\ + t\,)^{-\frac{}{2}} \ 1\ 1 \ k\ 1\ n\ k\ 1\ 2\ 2 \quad 2\ q- \quad cs\mathrm{d}t\mathrm{d}s \quad = \quad cl^k \int_0^1 \int_0^1 s^{-} t^{--} (l\ s\ + t\,)^{-\frac{n}{2}}$$

$\mathrm{d}t\mathrm{d}s$

$$= cl^q \quad n+k \quad 1\ 1 \quad k\ 1\ n\ k\ 1\ 2 \quad 2\ 2\ q- \quad - \int_0^1 \int_0^1 s^{-} t^{--} (s\ + l^{-} t\,)^{-\frac{n}{2}} \mathrm{d}t\mathrm{d}s.$$

The monotone convergence theorem gives

$$\lim_{l\to\infty} C_q(K_l, \mathrm{d}t\mathrm{d}s \quad \mathbb{S}^{n-1} \cap L) = \lim_{l\to\infty} \int_0^1 \int_0^1 s^{q-1} t^{n-k-1} (1 + l^{-2} t^2)^{\frac{q-n}{2}}$$

$$= \int_0^1 s^{q-1}\mathrm{d}s \cdot \int_0^1 t^{n-k-1}\mathrm{d}t = \frac{1}{q\,(n-k)}$$

and

$$\lim_{l\to\infty} C_q(K_l, S^{n-1}) = \lim_{l\to\infty} \int_0^1 \int_0^1 s^{k-1} t^{n-k-1} (s^2 + l^{-2} t^2)^{\frac{q-n}{2}} \mathrm{d}t\mathrm{d}s$$

$$= \int_0^1 s^{q-n+k-1}\mathrm{d}s \cdot \int_0^1 t^{n-k-1}\mathrm{d}t = \frac{1}{(q-n+k)(n-k)}.$$

Hence,

$$\lim_{l\to\infty} \frac{\widetilde{C}_q(K_l,\ \mathbb{S}^{n-1} \cap L)}{C_q(K_l, \mathbb{S}^{n-1})} = \frac{q-n+k}{q}. \qquad \square$$

5. The case $n < q < n+1$

The only remaining open range of q regarding the existence of a subspace bound on the qth dual curvature of symmetric convex bodies is when $q \in (n, n+1)$. It is apparent from the proof of Theorem 1.7 that an extension of Theorem 1.9 to $p \in (0, 1)$ would suffice. However, there are examples even in the 1-dimensional case showing that this is not possible.

Proposition 5.1. *Let $p \in (0, 1)$. There is an interval $K \subset \mathbb{R}$ and $\lambda \in [0, 1]$ such that for $K_\lambda = \lambda K + (1-\lambda)(-K)$ we have*

$$\int_{K_\lambda} |x|^p \,dx < |2\lambda - 1|^p \int_K |x|^p \,dx.$$

Proof. Let $\varepsilon > 0$, $K = [\varepsilon, \varepsilon + 1]$ and $\lambda = \frac{\varepsilon+1}{2\varepsilon+1} > \frac{1}{2}$ so that $K_\lambda = [0, 1]$. Then

$$(p+1) \int_{K_\lambda} |x|^p \,dx = 1$$

and

$$(p+1)(2\lambda-1)^p \int_K |x|^p \,dx = (p+1)(2\varepsilon+1)^{-p} \int_\varepsilon^{\varepsilon+1} x^p \,dx$$

$$= \frac{(\varepsilon+1)^{p+1} - \varepsilon^{p+1}}{(2\varepsilon+1)^p}.$$

Let $f(t) = (1+t)^{p+1} - t^{p+1}$ and $g(t) = (2t+1)^p$. Since

$$f(0) = 1 = g(0), f(0) = (p+1) > 2p = g(0),$$

we have

$$\frac{(\varepsilon+1)^{p+1} - \varepsilon^{p+1}}{(2\varepsilon+1)^p} > 1$$

for small ε and hence,

$$\int_{K_\lambda} |x|^p \,dx < (2\lambda-1)^p \int_K |x|^p \,dx.$$

Nevertheless, the examples given in the proof of Proposition 1.8 indicate that the subspace bound (1.10) might also be correct for $q \in (n, n+1)$. Next we will verify this in the special case of parallelotopes and to this end we need the next lemma about integrals of quasiconvex functions $f : \mathbb{R}^n \to \mathbb{R}$, i.e., functions whose sublevel sets

$$L^-_f(\alpha) = \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) \leq \alpha\}, \alpha \in \mathbb{R},$$

are convex. The lemma follows directly from [2, Theorem 1] where the reverse inequality is proved for quasiconcave functions and superlevel sets.

Lemmaconvex, compact5.2. *Let fsets.*: R ^{n}*Let*→ $K^{\mathbb{R}}_{\geq}$∈ K0 *be*n *anwith even,* K =*quasiconvex*−K *and* dim*function,*K = k*that. Let is* λ *integrable* ∈ [0, 1] *and on*

$\boldsymbol{v} \in \mathbb{R}^n$. *Then*

$$\int_{K+\lambda \boldsymbol{v}} f(\boldsymbol{x})\,d\mathcal{H}^k(\boldsymbol{x}) \leq \int_{K+\boldsymbol{v}} f(\boldsymbol{x})\,d\mathcal{H}^k(\boldsymbol{x}).$$

Moreover, equality holds for $\lambda \in [0, 1)$ *if and only if for every* $\alpha > 0$

$$(K + \boldsymbol{v}) \cap L^-_f(\alpha) = (K \cap L^-_f(\alpha)) + \boldsymbol{v}.$$

Proof. Apply [2, Theorem 1] to the function $\tilde{f}(\boldsymbol{x}) = \max(c - f(\boldsymbol{x}), 0)$ where $c = \sup\{f(\boldsymbol{x}+\boldsymbol{v}) : \boldsymbol{x} \in K\}$. □

Based on Lemma 5.2 we can easily get a lower bound on the subspace concentration of prisms.

Proposition 5.3. *Let* $\boldsymbol{u} \in \mathbb{S}^{n-1}$ *and let* $Q \in \mathcal{K}^n$ *with* $Q = -Q$, $Q \subset \boldsymbol{u}^\perp$ *and* $\dim Q = n-1$. *Let* $\boldsymbol{v} \in \mathbb{R}^n \setminus \operatorname{aff} Q$ *and let* P *be the prism* $\operatorname{conv}(Q - \boldsymbol{v}, Q + \boldsymbol{v})$. *Then for* $q > n$

$$\tilde{C}_q(P, \{\boldsymbol{u}, -\boldsymbol{u}\}) > \frac{1}{q}\tilde{C}_q(P, \mathbb{S}^{n-1}). \tag{5.1}$$

Proof. Let $L = \operatorname{lin}\{\boldsymbol{u}\}$. Since the dual curvature measure is homogeneous we may assume that $\boldsymbol{u}, \boldsymbol{v} = 1$. By Lemma 2.1 we may write

$$\tilde{C}_q(P, \mathbb{S}^{n-1}) = \frac{q}{n} \int_{P|L^\perp} \int_{P \cap (\boldsymbol{y}+L)} |\boldsymbol{z}|^{q-n}\, d\mathcal{H}^{n-1}(\boldsymbol{z})\, d\mathcal{H}(\boldsymbol{y})$$

1

$$= \frac{q}{n}\int_{-1}^{1}\int_{Q+\tau \boldsymbol{v}} |\boldsymbol{z}|^{q-n}\, d\mathcal{H}^{n-1}(\boldsymbol{z})\, d\tau$$

$$= 2\frac{q}{n}\int_{0}^{1}\int_{Q+\tau \boldsymbol{v}} |\boldsymbol{z}|^{q-n}\, d\mathcal{H}^{n-1}(\boldsymbol{z})\, d\tau.$$

Applying Lemma 5.2 to the inner integral gives

$$\tilde{C}_q(P,S^{n-1}) \le 2\frac{q}{n}\int_{0}^{1}\int_{Q+\boldsymbol{v}} |\boldsymbol{z}|^{q-n}\, d\mathcal{H}^{n-1}(\boldsymbol{z})\, d\tau$$

$$= 2\frac{q}{n}\int_{Q+\boldsymbol{v}} |\boldsymbol{z}|^{q-n}\, d\mathcal{H}^{n-1}(\boldsymbol{z}).$$

On the other hand,

$$\tilde{C}_q(P,\{\boldsymbol{u},-\boldsymbol{u}\}) = 2\frac{q}{n}\int_{0}^{1}\int_{\tau(Q+\boldsymbol{v})} |\boldsymbol{z}|^{q-n}\, d\mathcal{H}^{n-1}(\boldsymbol{z})\, d\tau$$

$$= 2\frac{q}{n}\int_{0}^{1}\int_{Q+\boldsymbol{v}} \tau^{q-n}|\boldsymbol{z}|^{q-n}\cdot \tau^{n-1}\, d\mathcal{H}^{n-1}(\boldsymbol{z})\, d\tau$$

$$= 2\frac{q}{n}\int_{Q+\boldsymbol{v}} |\boldsymbol{z}|^{q-n}\, d\mathcal{H}^{n-1}(\boldsymbol{z}) \int_{0}^{1}\tau^{q-1}\, d\tau$$

$$= 2\frac{1}{n}\int_{Q+\boldsymbol{v}} |\boldsymbol{z}|^{q-n}\, d\mathcal{H}^{n-1}(\boldsymbol{z}).$$

This gives (5.1) without strict inequality. Suppose we have equality. Then the equality characterization of Lemma 5.2 implies

$$(Q+\boldsymbol{v}) \cap rB_n = (Q \cap rB_n) + \boldsymbol{v}$$

for almost all $r > 0$. But for small r the left hand side is empty and hence equality in (5.1) cannot be attained.

As a consequence we deduce an upper bound on the subspace concentration of dual curvature measures of parallelotopes.

Corollary 5.4. *Let $P \in K_e^n$ be a parallelotope, and let $L \subset R^n$ be a proper subspace of R^n. Then for $q > n$*

$$\frac{\tilde{C}_q(P, S^{n-1} \cap L)}{\tilde{C}_q(P, S^{n-1})} < \frac{q - n + \dim L}{q}.$$

Proof. Let $\pm u_1, \ldots, \pm u_n \in S^{n-1}$ be the outer normal vectors of P. In particular, $u_1, \ldots, u_n$ are linearly independent and by Proposition 5.3

$$\tilde{C}_q(P, S^{n-1} \cap L) = \sum_{u_i \in L} \tilde{C}_q(P, \{u_i, -u_i\})$$

$$= \tilde{C}_q(P, S^{n-1}) - \sum_{u_i \notin L} \tilde{C}_q(P, \{u_i, -u_i\})$$

$$< \tilde{C}_q(P, S^{n-1}) - \sum_{u_i \notin L} \frac{1}{q} \tilde{C}_q(P, S^{n-1}).$$

L can contain at most $\dim L$ of the u_i's and therefore

$$\tilde{C}_q(P, S^{n-1} \cap L) < \tilde{C}_q(P, S^{n-1}) - \frac{n - \dim L}{q} \tilde{C}_q(P, S^{n-1}) = \frac{q - n + \dim L}{q} \tilde{C}_q(P, S^{n-1}).$$

In particular this settles the 2-dimensional case as it can be reduced to proving a subspace bound for parallelograms.

Proof of Theorem 1.11. Let $L = \mathrm{lin}\{u\}$, $u \in S^1$. If $F = K \cap H(u, h_K(u))$ is a singleton, inequality (1.12) trivially holds. Assume $\dim F = 1$. By an inclusion argument it suffices to prove (1.12) for $P = \mathrm{conv}(F \cup (-F))$. Since F is a line segment, P is a parallelogram and Corollary 5.4 gives (1.12) for P. □

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