RESEARCH ARTICLE OPEN ACCESS

A Numerically Robust T-Matrix Method for Multiple Polygons

K.Rekha [1], Dr.K.Thirugnanasambandam [2]
Phd Scholar [1], Assistant Professor [2]
Department of Mathamatics
Manonmanium Sundaranar University, Tirunelveli [1]
Muthurangam Government Arts College, Vellore [2]
Tami Nadu - India

ABSTRACT

Here we introduce the Transmission Matrix (T-matrix) T-matrix methods for single and multiple obstacles (1.2), for which the results and derivations. We focus in particular on a relatively recent approach. (The novel contribution of this paper is the combination with the Embedding Formulae which significantly reduces the computational cost required. Hence, it should be noted that the derivations and results are not new. In this paper we emphasis on the T-matrix system for single obstacles. The building of the T-matrix will be undistinguishable for the numerous scattering invention.

Keywords:- T-matrix

I. INTRODUCTION

1.1 T-matrix methods for single scattering

Initially we focus on the T-matrix method for single obstacles. The construction of the T-matrix will be identical for the multiple scattering formulation of the problem of 1.2, in which each obstacle will have a single T-matrix, independent of the incident field. We are interested in the T-matrix because it extends easily to multiple scattering problems. The single scattering method has other applications, in particular for modelling moving obstacles. We do not explore this here.

1.1.1 Specific problem statement

Here we assume that the origin lies inside of a bounded open set $\Omega_- \subset R^2$ with boundary $\partial\Omega$, such that the ball B_{R-} with radius $R_- \geq \mathrm{diam}(\Omega_-)/2$, is centred at the origin and contains Ω_-. We do not impose the requirement that Ω_- is connected, i.e., this obstacle can consist of many obstacles, and all of the following will still hold, provided a method to approximate the problem on Ω_- is available. For the purpose of understanding the T-matrix method, it is simpler to consider Ω_- as a single connected set. We consider the expansion of the incident field in terms of *regular wavefunctions* ψ_ℓ^{inc}, that is $u^{\mathrm{inc}}(\mathbf{x}) = Xb_\ell\psi_\ell^{\mathrm{inc}}(\mathbf{x})$ in B_-, where $\psi_\ell^{\mathrm{inc}}(\mathbf{x}):=J_{|\ell|}(k|\mathbf{x}|)e^{i\ell\theta_\mathbf{x}}$,

(1.1) ℓ where J_n is the Bessel function of the first kind order n and $\theta_\mathbf{x} \in [0,2\pi)$ is the angle that $\mathbf{x} \in R^2$ makes with the x_1-axis. We expand the scattered field in terms of *radiating wavefunctions* ψ_ℓ^s, that is $u^s(\mathbf{x}) = Xa_\ell\psi_\ell^s(\mathbf{x})$ in $R^2 \setminus B_-$ where $\psi_\ell^s(\mathbf{x}) := H^{(1)}_{|\ell|}(k|\mathbf{x}|)e^{i\ell\theta_\mathbf{x}}$, (1.2) ℓ where $H_n^{(1)}$ is the Hankel function of the first kind, order n. At this stage, we consider the infinite dimensional case, for which both sums (1.1) and (1.2) are over all $\ell \in Z$. The T-matrix is the matrix T that maps $\mathbf{a} := (a_\ell)_\ell$ to $\mathbf{b} := (b_\ell)_\ell$, hence

$$T\mathbf{a} = \mathbf{b}. \tag{1.3}$$

For incident fields such as plane wave and point source incidence, the coefficients b_ℓ are known and can be written explicitly (see [24]).

1.1.2 Computing the entries of the T-matrix

Given (1.3) and the coefficients $\mathbf{a}$, if we can compute (and invert) T then we have a representation for the scattered field from (1.2). The original formulation of T-matrix of [55] contains two methods to compute T, via the representation

$T = -BA^{-1}$.

(1.4)

The first method requires

$$(A)_{mn} = \frac{\mathrm{i}}{4}\left(\partial_{\mathbf{n}}^{+}\psi_n^{\mathrm{inc}}, \psi_m\right)_{L^2(\partial\Omega)}, \quad \text{and} \quad (B)_{mn} = \frac{\mathrm{i}}{4}\left(\partial_{\mathbf{n}}^{+}\psi_n^{\mathrm{inc}}, \psi_m^{\mathrm{inc}}\right)_{L^2(\partial\Omega)}, \qquad (1.5)$$

whilst the second approach takes

$$(A)_{mn} = \frac{\mathrm{i}}{4}\left(\psi_n^{\mathrm{inc}}, \partial_{\mathbf{n}}^{+}\psi_m\right)_{L^2(\partial\Omega)}, \quad \text{and} \quad (B)_{mn} = \frac{\mathrm{i}}{4}\left(\psi_n^{\mathrm{inc}}, \partial_{\mathbf{n}}^{+}\psi_m^{\mathrm{inc}}\right)_{L^2(\partial\Omega)}. \qquad (1.6)$$

Both approaches are commonly referred to as the *Null Field Method*, which along with other approaches (not mentioned here), can become numerically unstable for certain geometries Ω_- due to the singular nature of the Hankel functions (for more details see [23, 3]). This motivated the Tmatrom method of [21], for which

$$T_{mn} = \frac{1}{4}\frac{k}{\pi}\mathrm{i}^{|m|}(1+\mathrm{i})\int_0^{2\pi}[\mathcal{F}_\infty\psi_n^{\mathrm{inc}}](\theta)\mathrm{e}^{-\mathrm{i}m\theta}\,\mathrm{d}\theta, \qquad (1.1)$$

where F_∞ denotes the far-field map of (1.16). It follows that the Tmatrom method does not suffer from the same stability issues as other T-matrix methods, since the integrand of (1.1) is smooth. Tmatrom has the additional requirement that a *solver*, by which we loosely mean a numerical method which maps the Regular Wavefunction ψ_n^{inc} to an approximation of the far-field pattern $F_\infty\psi_n^{\mathrm{inc}}$, must be incorporated into the Tmatrom method. A suitable solver may involve the space $V_N^{\mathrm{HNA}}(\partial\Omega)$ as introduced in Paper 2, although there are many suitable choices. In the numerical example that follows we use MPSpack of [4].

1.1.3 Truncation of the T-matrix

In practice, we must truncate the T-matrix so that it is finite dimensional, summing over indices $\ell = -\hat{N}$ to $\ell = \hat{N}$, for $\hat{N} \in \mathbb{N}_0$. The finite dimensional case with truncated T results in an approximation to u^s via (1.2) and (1.3), and as $\hat{N}$ increases, this approximation improves. We define the truncated T-matrix as

$$\hat{T} := (T)_{n,m=-\hat{N}}^{\hat{N}} \in \mathbb{C}^{(2\hat{N}+1)\times(2\hat{N}+1)}, \qquad \text{for } \hat{N} \in \mathbb{N}_0.$$

The number of dimensions $\hat{N}$ is typically chosen to satisfy the condition of [56]:

$$\hat{N} = \lceil kR_- + 4(kR_-)^{1/3} + 5 \rceil, \qquad (1.8)$$

which is justified for the case (1.1) with point source or plane wave incidence in [24, Theorems 3.6 and 3.1]. This is another advantage over the null field method, which does not have this theoretical validation. Given that we sum over negative and positive indices of the wavefunctions, we require the far-field pattern and hence the solution, of $2\hat{N} + 1$ problems with different radiating wavefunction incidence. It is clear from (1.8) that $k \lesssim \hat{N}$ as $k \to \infty$, hence the number of solves required by the Tmatrom method grows more than linearly with the wavenumber k, posing potential difficulties at large wavenumbers.

Given the truncated vector of coefficients

$$\hat{\mathbf{a}} := \hat{T}^{-1}(b_\ell)_{\ell=-\hat{N}}^{\hat{N}},$$

we can construct an approximation to the far-field pattern (1.16), by expanding each term in the truncated series (1.2) as $r \to \infty$ using [18, (10.2.5)],

$$u^\infty(\theta) \approx \hat{u}^\infty_{\hat{N}}(\theta) := \sum_{\ell=-\hat{N}}^{\hat{N}} \mathrm{i}^{-|\ell|-1}\hat{a}_\ell \mathrm{e}^{\mathrm{i}(\ell\theta)}. \qquad (1.9)$$

We have the following error estimate from [24, Theorem 3.9].

THEOREM 1.1. *For scattering of a plane wave by a single obstacle Ω_-, if $\hat{N} > kR_-/2 + 1$ then the following error bound holds:*

$$\|u^\infty - u^\infty_{\hat{N}}\|^2_{L^2(0,2\pi)} \le C\hat{N}^2\left(\frac{R_- k\mathrm{e}}{2\hat{N}}\right)^{2\hat{N}} + C'\epsilon_F^2, \qquad (1.10)$$

*where C and C′ are positive constants independent of k and N*ˆ, ϱ_F *denotes the error in the far-field approximation of the solver used, and* $\hat{u}^{\infty}_{\hat{N}}$ *is the approximation* (1.9) *to the far-field pattern* u^{∞} *of* (1.16).

1.2 T-matrices for multiple scattering

In this section we outline the procedure to extend any T-matrix method to multiple obstacles, which is based on the derivation of [22, 2.2]. Everything in this section holds for the infinite dimensional or truncated T-matrix case. Suppose now that Ω_- consists of n_γ pairwise disjoint obstacles, which we denote Ω_i for $i = 1,\ldots,n_\gamma$, hence $\Omega_- = \cup_{i=1}^{n_\gamma}\Omega_i$. For each obstacle, we denote by $\mathbf{x}^c_i$ a point inside of Ω_i, and denote by $T_{(i)}$ the T-matrix corresponding to the obstacle Ω_i with a coordinate system translated by $-\mathbf{x}^c_i$ in each case, so that the origin is inside of the obstacle. We impose the additional constraint that there exists a collection of pairwise disjoint balls $B_{R_i}(\mathbf{x}^c_i) \supset \Omega_i$, $R_i > 0$ for $i = 1,\ldots,n_\gamma$. Recalling the example (4.1), we now formulate the multiple scattering T-matrix method by considering a single scattering problem on each obstacle, where the sum of the scattered fields emanating from all other obstacles is absorbed into the incident field of a single scattering T-matrix problem on the *i*th obstacle Ω_i:

uinci := *uinc* + X′ *usi′*, in *Bi* (1.11)

i =6 *i*

where $u^s_{i'}$ is the (also unknown at this stage) contribution to the scattered field from the obstacle i'. Recalling that each single scattering problem requires the origin to be positioned inside of the scatterer, we will make use of the *Translation Addition Theorem* of [19] to translate $\mathbf{x}^c_{i'}$ to $\mathbf{x}^c_i$. Proceeding as in [22, 2.2], we define the two-dimensional analogue of translation addition matrix of [19, (51)] as

$$(S^{(i'\to i)})_{\ell,\ell'} := w_{\ell,\ell'}\rho^{(i')}_{\ell,\ell'}\mathrm{e}^{\mathrm{i}(\ell-\ell')\theta(\mathbf{x}^c_{i'}-\mathbf{x}^c_i)}H^{(1)}_{|\ell-\ell'|}(k|\mathbf{x}^c_{i'}-\mathbf{x}^c_i|)w_{\ell',\ell}$$

where

$$\rho^{(i')}_{\ell,\ell'} := \frac{\sqrt{\pi k}}{(-\mathrm{i})^{|\ell|(1-\mathrm{i})}H^{(1)}_{\ell'}(kR_{i'})}$$

and

$$w_{\ell',\ell} := \frac{1}{\sqrt{2\pi}}(-\mathrm{i})^{|\ell|-|\ell'|-|\ell-\ell'|}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(|\ell|-|\ell'|-|\ell-\ell'|)\theta}\,\mathrm{d}\theta.$$

Denote by $\mathbf{a}_{(i)}$ the vector $\mathbf{a}$ of (1.1), corresponding to the solution to the single obstacle problem (1.1.1) on Ω_i. We seek to determine the vector $\mathbf{b}_{(i)}$, which corresponds to the field scattered by Ω_i, given that additional terms have been absorbed into the incident field (1.11), emanating from the other scatterers. We may expand the terms in (1.11) to obtain a representation for the incidence

$$u^{\mathrm{inc}}_i(\mathbf{x}) = \sum_\ell(\mathbf{b}_{(i)})_\ell\psi^{\mathrm{inc}}_\ell(\mathbf{x}) + \sum_{i'\neq i}\sum_\ell(S_{(i'\to i)}\mathbf{a}_{(i')})_\ell\psi^s_\ell(\mathbf{x}-\mathbf{x}^c_{i'}), \quad \mathbf{x}\in B_i. \qquad (1.12)$$

Now multiplying (1.12) by $T_{(i)}$, the T-matrix for the obstacle Ω_i, we obtain the contribution to the scattered field from Ω_i,

$$u^s_{(i)}(\mathbf{x}) = \sum_\ell(T_{(i)}\mathbf{b}_{(i)})_\ell\psi^s_\ell(\mathbf{x}) + \sum_{i'\neq i}\sum_\ell(T_{(i)}S_{(i'\to i)}\mathbf{a}_{(i')})_\ell\psi^s_\ell(\mathbf{x}-\mathbf{x}^c_{i'}), \quad \mathbf{x} \text{ in } \mathbb{R}^2\setminus B_i, \qquad (1.13)$$

hence to determine the coefficients $\mathbf{b}_{(i)}$ of the scattered field for each Ω_i, the system to solve is

$$\mathbf{a}_{(i)} - \sum_{i\neq i'}T_{(i)}S_{(i'\to i)}\mathbf{a}_{(i')} = T_{(i)}\mathbf{b}_{(i')}, \quad \text{for } i = 1,\ldots,n_\gamma. \qquad (1.14)$$

As in the single scattering case, in practice each T-matrix (and consequently each translation addition matrix $S_{(i'\to i)}$) must be truncated in accordance with (1.8).

II. REDUCING THE NUMBER OF SOLVES REQUIRED

Here we extend the Tmatrom method by combining it with the Embedding Formulae used in Paper 6. The theory here is for a single obstacle, but is equally adaptable to multiple obstacles using the ideas discussed in 1.2. We suppose now that our obstacle Ω_- is a rational (in the sense of 6.1) polygon. First we introduce the Herglotz kernel (of Definition 1.8) for the far-field pattern of the ℓth Regular Wavefunction ψ_ℓ^{inc}, which follows by the Jacobi-Anger expansion (e.g. [11, (3.89)]) and the definition (1.1):

(1.15)
$$g_\ell(\theta) := \begin{cases} \mathrm{e}^{-\mathrm{i}\ell\theta}/(2\pi\mathrm{i}^\ell), & \ell \geq 0, \\ (-1)^\ell \mathrm{e}^{-\mathrm{i}\ell\theta}/(2\pi\mathrm{i}^\ell), & \ell < 0 \end{cases}$$
,

hence we can write (as in 6.4.1)

$$\psi_\ell^{\text{inc}}(\mathbf{x}) = \int_0^{2\pi} g_\ell(\alpha) u_{PW}^{\text{inc}}(\mathbf{x}; \alpha)\, \mathrm{d}\alpha.$$

Using this, we may use our HNA method for Herglotz type incidence of 3.1 with Herglotz kernel g_ℓ, and solve for $\ell = -M_T, ..., M_T$. Using results computed in this thesis, we can determine the error in the approximation.

COROLLARY 1.2. *Suppose the conditions of Theorem 1.1 are satisfied, and the Herglotz-type HNA method of 3.1 is chosen as the solver to be used in conjunction with Tmatrom. Then the constant ϱ_F of Theorem 1.10 is bounded by*
$\varrho F \leq Ck-1/2Cq(k)M\infty(u)J(k)e-p\tau\Gamma$,
where C,J,p and τ_Γ are the constants from Corollary 2.11, C_q is the stability constant from Remark 2.13, whilst

$$M_\infty(u) \leq \left(\sqrt{2\pi} + 2C_1\sqrt{\pi}|\Gamma|^{1/2} \left[_{2\mathrm{diam}(\Omega} \_) + \frac{1}{2k} \right] k^{1/2} \log^{1/2}(2 + kL_*) \right)$$
,

where C_1 and L_* are as in Theorem 3.1.

Proof. Follows immediately from (2.12), whilst the bound on $M_\infty(u)$ follows from Theorem 3.3, noting that the required bound on the Herglotz kernel is $\mathrm{k}g_\ell \mathrm{k}L2_{(0,2\pi)} = 1$, by (1.15).

When solving high frequency problems, Tmatrom with Herglotz-type HNA clearly provides a numerically robust approximation, with explicit error bounds available in the specific case of single scattering by a plane wave. The key advantage is that once the T-matrix has been computed, it can be re-used for different incident waves, and problems can be solved very quickly, as the coefficients are given explicitly. This is exactly the same benefit of using the Embedding Formulae in Paper 6; once the canonical problems were solved, we can solve easily for any incident angle, and bound the error in doing so. We do not compare the efficiency of the two methods here, instead we combine them.

We now provide a brief example to motivate integration of the Embedding Formulae with a numerical solver, before incorporating with Tmatrom. Suppose that we are solving the problem of scattering of a plane wave with wavenumber $k = 1000$ by multiple (identical) squares, of identical orientation. The same T-matrix may be used for each square. Given that the total number of solves is $2M_T+1$, and we must satisfy the condition (1.8), we must solve for $\ell \in \{-1045,...,1045\}$ incident fields, a total of 2091 solves. Although it is only necessary to recompute the right-hand side of the Galerkin system (2.11) in each instance, as k grows the number of solves grows faster than $2k$, clearly introducing a k-dependence to the method. However, from Remark 6.1, one can implement an embedding formula for a square by solving for only eight plane waves, after which we can use the Embedding Formulae to produce the far-field pattern for Herglotz kernel g_ℓ of 1.15 for $\ell \in \{-1045,...,1045\}$, enabling us to compute the T-matrix without the need for further solves. This number does not increase with frequency, and depends only on the geometry of the obstacle.

We may write this idea generally for any rational (in the sense of 6.1) polygonal obstacle Ω_-. Using embedding theory

$$\begin{aligned} \mathcal{F}_\infty \psi_\ell^{\text{inc}}(\theta) &= \int_0^{2\pi} g_\ell(\alpha) \mathcal{F}_\infty \left[\mathrm{e}^{\mathrm{i}k(\cdot)\cdot\mathbf{d}_\alpha} \right](\theta)\, \mathrm{d}\alpha, \\ &= \int_0^{2\pi} g_\ell(\theta) D(\theta, \alpha)\, \mathrm{d}\alpha, \end{aligned}$$
where $\mathbf{d}_\alpha := (\cos\alpha, -\sin\alpha)$

where $F\infty$ is the far-field map (1.11). Inserting into (1.1) and substituting (6.4) we obtain an extension to the Tmatrom method, for which the entries are computed via

$$T_{j\ell} = \frac{1}{4}\frac{k}{\pi}\mathrm{i}^{|j|}(1+\mathrm{i}) \int_0^{2\pi} \int_0^{2\pi} g_\ell(\alpha) \frac{\sum_{m=1}^{M_\omega} B_m(\alpha)\Lambda(\theta, \alpha_m) D(\theta, \alpha_m)}{\Lambda(\theta, \alpha)} \mathrm{e}^{-\mathrm{i}j\theta}\, \mathrm{d}\alpha\, \mathrm{d}\theta. \quad (7.16)$$

Hence only M_ω solves are required for the Tmatrom algorithm, where M_ω depends only on the geometry of Ω_-. In 6.1.2 it was shown that the representation (6.4) breaks down when implemented numerically, hence in practice the matrix entries should be computed using

$$T_{j\ell} = \frac{1}{4}\frac{k}{\pi}\mathrm{i}^{|j|}(1+\mathrm{i}) \int_0^{2\pi} \int_0^{2\pi} g_\ell(\alpha) \mathcal{E}_{\mathcal{P}_F}^{\circledast} D(\theta, \alpha; N_T) \Lambda(\theta, \alpha) \mathrm{e}^{-\mathrm{i}j\theta}\, \mathrm{d}\alpha\, \mathrm{d}\theta, \quad (1.11)$$

where $\mathcal{E}^{\circledast}_{\mathcal{P}_F} D(\theta, \alpha; N_T)$ is the Combined Expansion Approximation of Definition 6.6, with N_T the parameter corresponding to the degree of the Taylor expansion taken. The integral (1.11) must be computed using a quadrature rule, as discussed in Appendix B. Where possible, these points should be chosen in the same spirit as (6.34), with α nodes as far as possible from the points in $[0,2\pi)$ at which only a first order Taylor approximation is used.

Tmatrom is in some ways superior to traditional T-matrix methods (such as the Null Field approaches (1.5) and (1.6)), in that it is provably stable for any configuration. This comes at the cost of the requirement to solve $O(k)$ scattering problems before the T-matrix can be computed, which is not a requirement of the other (less stable) T-matrix methods. By coupling Tmatrom with the Embedding Formulae of Paper 6, this $O(k)$ dependence becomes $O(1)$, and the cost of computing the more stable Tmatrom T-matrix has the same k-dependence as a traditional T-matrix method, such as (1.5) or (1.6). Therefore this combined approach offers stability, at no extra k-dependent cost. Justifying this claim with numerical results is a key area for future work. Figure 1.1 shows the output of the combination of Tmatrom with our embedding solver and the MPSpack solver. This required 8, 12 and 30 solves on the triangle, square and pentagon respectively, a total of 50 solves, a number independent of k. For wavenumber $k = 5$, using MPSpack without solving via the Embedding Formulae results in a total of 21 solves on each scatterer, hence a total of 81 solves. So even at a relatively low wavenumber using a non-HNA solver, the Embedding Formulae can reduce the number of solves required.

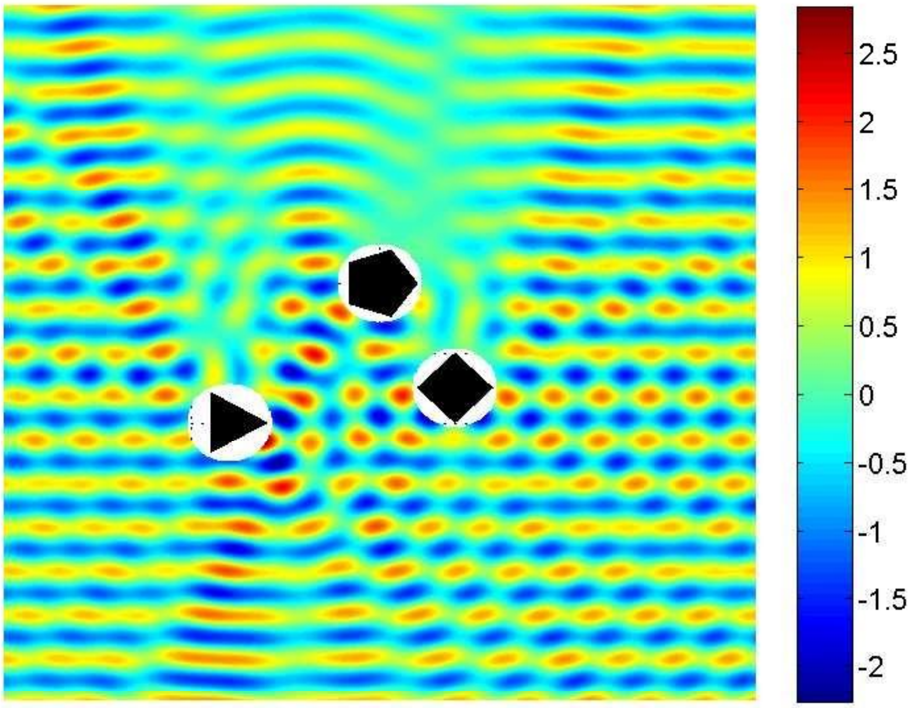

Figure 1.1: Real part of total field u for a configuration of multiple polygons, with incident field $u^{\mathrm{inc}}_{PW}(\cdot;\pi/2)$ solved using MPSpack as the solver used for the embedding implementation, which in turn is used as the solver for Tmatrom. The expansion is only valid outside of the balls B_i containing the obstacle Ω_i, for $i = 1,2,3$.

III. CONCLUSION

We conclude this paper with a summary of the k-dependency of different Tmatrix methods. We first note that the matrix T is in principle the same regardless of if a null-field method (1.5), (1.6) or Tmatrom method (1.1) is used, hence it is recommended that the size of the truncated T-matrix grows like $O(k)$ (specifically (1.8)) for any approach. For cases where they are robust, traditional T-matrix methods may be performed relatively quickly at low k, requiring two integrals for each entry of the T-matrix ((1.5) and (1.6)). This suggests an advantage over Tmatrom, which (i) requires a new scattering problem to be solved for each of the $O(k)$ columns of T, whilst (ii) each column in turn may require $O(k)$ such integrals if (for example) a standard Galerkin BEM solver is used. However, for polygons an HNA solver may be used to reduce the dependency of the Galerkin system to $O(1)$ (as discussed in 2.3) which overcomes (ii), and we have shown that Embedding formulae may be used to reduce the number of scattering problems to $O(1)$, which overcomes (i). Moreover, for large k and hence large T-matrices, numerical cost of the inverted matrix of (1.4) will increase for standard T-matrix methods, whilst Tmatrom does not require the inversion of any large matrices, so at large k a method combining HNA BEM and Embedding formulae with Tmatrom is advantageous over a standard T-matrix method. Investigating this combination is a key area for future work; the only k-dependence with such a method would take the form of $O(k^2)$ integrals used to compute T, which contain smooth integrands, and may be computed using sophisticated quadrature routines . The purpose of this paper was to lay the ground work for such an implementation.

REFERENCES

[1] Jean-Claude N´ed´elec. *Acoustic and electromagnetic equations*, volume 144 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2001. Integral representations for harmonic problems.

[2] S. Nonnenmacher and M. Zworski. Quantum decay rates in chaotic scattering. *Acta. Math.*, 203(2):149–233, 2009.

[3] E. Parolin. A hybrid numerical-asymptotic boundary element method for highfrequency wave scattering. Master's thesis, University of Oxford, 2015.

[4] A. Quarteroni and A. Valli. *Numerical approximation of partial differential equations*, volume 21. Springer Science & Business Media, 2008.

[5] S. A. Sauter and C. Schwab. *Boundary element methods*. Springer, 2010.

[6] E. A. Spence. Wavenumber-explicit bounds in time-harmonic acoustic scattering. *SIAM J. Math. Anal.*, 46(4):2987–3024, 2014.

[7] E. A. Spence. *"When all else fails, integrate by parts" - an overview of new and old variational formulations for linear elliptic PDEs*, paper 6. SIAM, 2015. Author's personal copy available at: http://people.bath.ac.uk/eas25/ibps.pdf, any numbered theorems, equations etc. inside of this thesis are referring to this version.

[8] E. A. Spence, S. N. Chandler-Wilde, I. G. Graham, and V. P. Smyshlyaev. A new frequency-uniform coercive boundary integral equation for acoustic scattering. *Comm. Pure Appl. Math.*, 64(10):1384–1415, 2011.

[9] E. A. Spence, I. V. Kamotski, and V. P. Smyshlyaev. Coercivity of combined boundary integral equations in high-frequency scattering. *Communications on Pure and Applied Mathematics*, 68(9):1587–1639, 2015.

[10] O. Steinbach. *Numerical approximation methods for elliptic boundary value problems: finite and boundary elements*. Springer Science & Business Media, 2007.

[11] P. C. Waterman. New formulation of acoustic scattering. *J. Acoust. Soc. Am.*, 45(6):1417–1429, 1969.

[12] W. J. Wiscombe. Improved Mie scattering algorithms. *Appl. Optics*, 19(9):1505– 1509, 1980.

[13] Jared Wunsch. Resolvent estimates with mild trapping. *J. E. D. P.´* , pages 1–15, 2012.